HW 1: solution sketches

(1) True or False? Every integer $n > 2$ occurs in some Pythagorean Triple. (Justify your answer.)

TRUE.

Every primitive Pythagorean Triple has the form $(a, b, c) = (r^2 - s^2, 2rs, r^2 + s^2)$ where $r > s > 0$, $\gcd(r, s) = 1$ and r and s have opposite parity (meaning that one is even and the other is odd). Moreover, any triple of the form $(r^2 - s^2, 2rs, r^2 + s^2)$ with $r > s > 0$ is a Pythagorean Triple, even if $\gcd(r, s) \neq 1$ or r and s have the same parity. (This statement means only that each term in the triple $(r^2 - s^2, 2rs, r^2 + s^2)$ is positive and $(r^2 - s^2)^2 + (2rs)^2 = (r^2 + s^2)^2$.)

To show that any $n > 2$ appears in a triple of the form $(r^2 - s^2, 2rs, r^2 + s^2)$ it suffices to note that if $n = 2k$ is even, then we can take $r = k$ and $s = 1$. Then $n = 2k = 2rs$, appears as the middle term of $(r^2 - s^2, 2rs, r^2 + s^2)$ and $r > s > 0$.

Now if $n > 2$ is odd, we want to arrange that $n = r^2 - s^2$ for some $r > s$. That is, we want $n = r^2 - s^2 = (r + s)(r - s)$. Using the fact that n is odd, and equating factorizations $n = n \cdot 1 = (r+s)(r-s)$, we can take $n = r+s$ and $1 = r-s$, solve for r and s, and obtain $r = (n+1)/2 \in \mathbb{Z}$ and $s = (n-1)/2 \in \mathbb{Z}$. Thus, for $r = (n+1)/2$ and $s = (n - 1)/2$ we have that $r > s > 0$ and the first term in $(r^2 - s^2, 2rs, r^2 + s^2)$ is n.

(2) Give a geometric proof that $\sqrt{3}$ is irrational. (Hint: It might be easier to show that $1 + \sqrt{3}$ is irrational, then deduce that $\sqrt{3}$ is also irrational.)

Start with a rectangle of height 1 and base $1+\sqrt{3}$. Delete two maximal subsquares (which are 1×1 squares) to obtain a remainder rectangle of base $(1 + \sqrt{3}) - 2$ (which is less than 1) and height 1. Now delete one maximal subsquare of base $(1 + \sqrt{3}) - 2 = \sqrt{3} - 1$ to obtain a remainder rectangle of base $\sqrt{3} - 1$ and height $1 - (\sqrt{3} - 1) = 2 - \sqrt{3}$. The current rectangle is similar to the starting rectangle, since

$$(1 + \sqrt{3})/1 = (\sqrt{3} - 1)/(2 - \sqrt{3}).$$

This implies that if you keep deleting maximal subsquares, the process will not terminate, so the starting values 1 and $1+\sqrt{3}$ are not commensurable. Equivalently, $1 + \sqrt{3}$ is not rational. Since 1 is rational, and the sum of rationals is rational, it follows that $\sqrt{3}$ cannot be rational either.

(3) Use the Euclidean algorithm to find an integral solution to $270x + 168y = 6$.

First use the Euclidean algorithm:

$$\begin{aligned}
270 &= 168 \cdot 1 + 102 \\
168 &= 102 \cdot 1 + 66 \\
102 &= 66 \cdot 1 + 36 \\
66 &= 36 \cdot 1 + 30 \\
36 &= 30 \cdot 1 + 6 \\
30 &= 6 \cdot 5 + 0
\end{aligned}$$

Rewriting this data yields $6 = (36 - 30 \cdot 1) = (36 - (66 - 36 \cdot 1) \cdot 1) = (36 \cdot 2 - 66 \cdot 1) = \cdots = \underline{\mathbf{5}} \cdot 270 - \underline{\mathbf{8}} \cdot 168$.