HW 10: solution sketches

(1) Exercise 8.7.1 from the text.

The answer is: the lines $y = 0$ and $z = 0$.

To see this, observe that, when the axes that are crossing at the origin are the y-axis and the z-axis, and the line at infinity is the line with equation $x = 0$, then the homogenization of $yz = 1$ is $x^2 - yz = 0$. In this coordinate system the origin has coordinates $\begin{bmatrix}1\\0\\0\end{bmatrix}$. The points on the line $x = 0$ and the curve $x^2 = yz$ are $\begin{bmatrix}0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$. Hence the desired lines are (i) the one through $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$, which is $z = 0$, and (ii) the one through $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\end{bmatrix}$, which is $y = 0$.

(2) Exercise 8.7.2 from the text.

The homogenization of $y = x^3$, in the coordinate system where the origin is at the place where the x-axis crosses the y-axis, is $yz^2 = x^3$. Now change coordinates so that the origin is at the place where the x-axis crosses the z-axis. That is, change coordinates of each point from $\begin{bmatrix}a\\b\\c\end{bmatrix}$ to $\begin{bmatrix}a\\c\\b\end{bmatrix}$ (interchanging the roles of y and z). Now the equation of the curve is $zy^2 = x^3$. Dehomogenize (by setting $z = 1$) to obtain $y^2 = x^3$, which is the equation for the same curve with respect to the new coordinates.

This shows that $y = x^3$ and $y^2 = x^3$ define the same curve up to a permutation of coordinates.

(3) Find all points of intersection of the curves $y = x^2$ and $y = x^3$, and compute the intersection multiplicity at each point of intersection.

The points of intersection, in homogeneous coordinates, are

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

(In affine coordinates these points are $(0, 0)$, $(1, 1)$, and the point at infinity in the direction $(0, 1)$.)

(a) Intersection multiplicity at $P = (0,0)$:

$$\begin{aligned} I_P(y - x^2, y - x^3) &= I_P(y - x^2, x^2 - x^3) \\ &= 2I_P(y - x^2, x) + I_P(y - x^2, 1 - x) \\ &= 2I_P(y, x) + 0 = 2. \end{aligned}$$

(b) Intersection multiplicity at $P = (1,1)$:

$$\begin{aligned} I_P(y - x^2, y - x^3) &= I_P(y - x^2, x^2 - x^3) \\ &= 2I_P(y - x^2, x) + I_P(y - x^2, 1 - x) \\ &= 2 \cdot 0 + I_P(y - x^2 - x(1 - x), 1 - x) \\ &= I_P(y - x, 1 - x) = I_P(y - x - (1 - x), 1 - x) \\ &= I_P(y - 1, -(x - 1)) \\ &= I_P(y - 1, x - 1) + I_P(y - 1, -1) \\ &= 1 + 0 = 1. \end{aligned}$$

(c) Intersection multiplicity at $P =$ intersection point at infinity: first change coordinates as in Problem 2 so that the intersection point at infinity is moved to the origin, and the affine equations become $y = x^2$ and $y^2 = x^3$. Then use $P = (0,0)$ and

$$\begin{aligned} I_P(y - x^2, y^2 - x^3) &= I_P(y - x^2, yx^2 - x^3) \\ &= 2I_P(y - x^2, x) + I_P(y - x^2, y - x) \\ &= 2I_P(y, x) + I_P(y - x^2 - (y - x), y - x) \\ &= 2 + I_P(x - x^2, y - x) \\ &= 2 + I_P(x, y - x) + I_P(1 - x, y - x) \\ &= 2 + I_P(x, y) + 0 = 3. \end{aligned}$$