

Group Theory

Homework Assignment II

Read all of the statements of the theorems in Chapter 5, and the proofs of the interesting theorems.

CHAPTER

5.2

5.3

PROBLEMS

1 (I), 5, 6 (I), 9, 10 (II)

7

ADDITIONAL PROBLEMS

1. (II) Find all finite solvable groups which have the property that $\text{Fit}(G) \cong \mathbb{Z}_5$.

2. Explain why, if G is a finite solvable group satisfying $\text{Fit}(G) \cong D_4$, it must be that $G \cong D_4$.

3. (II) Explain why, if G is a finite solvable group satisfying $\text{Fit}(G) \cong Q_8$, it must be that $8 \leq |G| \leq 48$.

4. (III) An idempotent endomorphism ($\varphi^2 = \varphi$) is called a retraction; it is a proper retraction if $\varphi \neq \text{id}$. Show that if φ is a proper retraction of a finite group G , then $\ker(\varphi) \not\leq \Phi(G)$.

5. Can you prove this statement in fifteen words or less? *If G is finite and $G/\Phi(G)$ is nilpotent, then G is nilpotent.*

6. (III) Any ring has a Frattini subring (= the intersection of maximal subrings), and also a Frattini ideal (= the largest $I \triangleleft R$ such that $I \subseteq M$ for every maximal subring $M \leq R$).

(a) Show that these notions can be different. (Produce an example where the Frattini subring is not an ideal.)

(b) State and prove the nongenerator properties for the Frattini subring of R and for the Frattini ideal of R .

7. (IV) Let G be a finite, 2-step nilpotent, p -group.

- (a) Show that if p is odd, then G has an abelian group word, $w(x, y)$.
That is, if $x \oplus y := w(x, y)$, then $\langle G; x \oplus y \rangle$ is an abelian group.
- (b) Is the same assertion true if p is even?
- (c) Is the assertion true if G is a 3-step nilpotent p -group?

8. (IV) For which $n \geq 0$ is there a finite group with exactly n nonnormal subgroups?

9. Let G be a finite group and let p be prime. Show that if p divides $|\Phi(G)|$, then p divides $[G : \Phi(G)]$. (You may refer to Theorem 9.1.2, if it helps.)

Collaboration Groups.

- (I) DuBeau, Gossett
- (II) Gerics, Lotfi, Ornstein
- (III) Meredith, Wang
- (IV) Ng, Wheeler