Group Theory
Homework Assignment I

Read all of the statements of the theorems in Chapter 5, and the proofs of the interesting theorems.

CHAPTER	PROBLEMS
5.1	**6 (I), 9, 11 (I), 12 (II), 13 (II)**

ADDITIONAL PROBLEMS

1. Give examples that distinguish between the subgroup properties *normal*, *characteristic*, *fully invariant*, and *verbal.* Explain why a subgroup of a free group is fully invariant iff it is verbal.

2. (III) Give an example of a marginal subgroup that is not fully invariant.

3. (III) Suppose that $\mathcal{W}_1$ is a "stronger" set of words than $\mathcal{W}_2$ in the sense that

$$\forall G(\forall w \in \mathcal{W}_1(G \models w = 1) \to \forall w \in \mathcal{W}_2(G \models w = 1)).$$

Show that for any group H

(a) $V_{\mathcal{W}_2}(H) \subseteq V_{\mathcal{W}_1}(H)$
(b) $M_{\mathcal{W}_1}(H) \subseteq M_{\mathcal{W}_2}(H)$

4. The n-th Engel word is the word

$$e_n(x, y) = [x, y, \dots, y], \quad n \ y\text{'s}.$$

An n-Engel group is a group satisfying the law $e_n(x, y) = 1$.

(a) Show that a group is 2-Engel (satisfies $[x, y, y] = 1$)) iff every 1-generated normal subgroup is abelian.
(b) Show that the following laws are consequences of $[x, y, y] = 1$:
  (i) $[x, y, z] = [y, z, x]$
  (ii) $[x, y, z]^3 = 1$
  (iii) $[x, y, z, t] = 1$

5. If $H \leq G$, show that $H \lhd G$ iff $[G, H] \subseteq H$.

6. (III) Show that if G is nilpotent and $a, b \in G$ have finite order, then the product ab also has finite order. (Hint: If $n = \mathrm{lcm}(|a|, |b|)$, then $a^n = b^n = 1$. Prove by induction on k that if G is nilpotent of class k, then $(ab)^{n^k} = 1$. For the induction argument, first consider $k = 1$. Then assume that the induction claim is true in $G/\gamma_k(G)$ and deduce it for G.)

7. (IV) Extend the ascending central series of a group G,

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \dots$$

transfinitely by taking unions at limit ordinals: $\zeta_\kappa(G) = \bigcup_{\lambda<\kappa} \zeta_\lambda(G)$. The largest term in this series is the *hypercenter* of G. G is *hypercentral* if it equals its hypercenter.

Show that if G is perfect, then the hypercenter of G is its center. (Hint: Use the Hall-Witt identity to show that if $H, K, L \lhd G$, then $[H, K, L] \leq [K, L, H][L, H, K]$. Then prove that $\zeta_2(G) = \zeta_1(G)$ by taking $(H, K, L) = (G, G, \zeta_2(G))$.)

8. (IV) Let G be a finite nonsolvable group. Let N be minimal among normal nonsolvable subgroups of G.

(a) Show that G has a normal subgroup, N_*, that is the largest for the property of being properly contained in N.

(b) Show that every normal subgroup $K \lhd G$ satisfies either (i) $N \subseteq K$ or (ii) $K \subseteq (N_* : N)$, but no normal subgroup of G satisfies both (i) and (ii). ($(N_* : N) = \{g \in G | [g, N] \subseteq N_*\}$.)

(c) Explain why G must be solvable if Norm(G) has no homomorphism onto the 2-element lattice.

Collaboration Groups.

(I) DuBeau, Gerics
(II) Gossett, Lotfi
(III) Meredith, Ng, Ornstein
(IV) Wang, Wheeler