Group Theory - HW3, Problem 8.3.11

Jordan DuBeau

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Problem. Let G be a simple group of order m and let p be a prime dividing m. If the class number of G exceeds $\frac{m}{p^2}$, then the Sylow p-subgroups of G are abelian.

Solution. We can write $m = d_1^2 + d_2^2 + \cdots + d_r^2$, where r is the class number of G and each d_k is the degree of an irreducible character χ_k. We may as well assume that $p^2|m$, because otherwise the p-Sylow subgroups of G have order p and must be abelian. So, if $r > \frac{m}{p^2}$, then we have $r \geq \frac{m}{p^2} + 1$. This implies that at least one nontrivial irreducible character has degree less than p. Otherwise, we would have

$$m = d_1^2 + d_2^2 + \cdots + d_r^2 \geq 1 + \overbrace{p^2 + \cdots + p^2}^{\text{at least } \frac{m}{p^2}} \geq 1 + \left(\frac{m}{p^2}\right) p^2 > m.$$

So, let χ be an irreducible character of G of degree less than p. Let ρ be the representation that affords χ.

Now, let P be a Sylow p-subgroup of G. Then $\rho \restriction P$ is not necessarily an irreducible representation of P, but it must be the direct sum of finitely many irreducible representations of P. Let $(\rho_i)_{i=1}^n$ be the irreducible representations of P whose direct sum is ρ, and let $(\chi_i)_{i=1}^n$ be the corresponding irreducible characters of P. Then, we have $\chi(1) = \sum_{i=1}^n \chi_i(1)$, and so it must be the case that $\chi_i(1) < \chi(1) < p$ for all i. As each $\chi_i(1)$ divides the order of P, we must have $\chi_i(1) = 1$ for all i. In other words, each χ_i is linear.

Now, we note that $\bigcap_{i=1}^n K_{\chi_i} \subseteq K_\chi$. This is because, if an element is mapped to the identity matrix by each ρ_i, then it must be mapped to the identity matrix by ρ, which is the direct sum of the ρ_i's. But K_χ is trivial, since it must be a proper normal subgroup of G and G is simple. So, by property (10) on the "Basic Properties of Characters of Finite Groups" handout, we can deduce:

$$[P, P] = \bigcap_{\hat{\chi} \text{ linear}} K_{\hat{\chi}} \subseteq \bigcap_{i=1}^n K_{\chi_i} \subseteq K_\chi = \{1\}.$$

This means that every commutator in P is trivial, and P is abelian.