Group Theory Assignment 3 Problem 6

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Proposition. *A finite nonabelian simple group has no irreducible representations of degree* 2.

Proof. Let $\rho : G \to \mathrm{GL}_2(\mathbb{C})$ be an irreducible representation of G with χ the character afforded by ρ and let χ_1 be the character afforded by the trivial representation. Since $\chi \in \mathrm{Irr}(G)$, we have that $\chi(1) \mid |G|$, so G has even order, and hence an element g of order 2. Let $\mathcal{B}$ be a basis for $\mathbb{C}^2$ in which $\rho(g)$ is diagonalized. Since $g = g^{-1}$, we have that $\rho(g)_{\mathcal{B}} = \overline{\rho(g)_{\mathcal{B}}}$ where $\rho(g)_{\mathcal{B}}$ is the matrix representation of $\rho(g)$ with respect to the basis $\mathcal{B}$. That is, the eigenvalues of $\rho(g)$ must be real roots of unity, either $+1$, -1, or both. By results about simple nonabelian groups proven in class, we know that the 2-Sylow subgroups of G have order 2, so $\langle g \rangle$ is a 2-Sylow subgroup. Further, since χ is a degree 2 representation, we must have $\chi(\langle g \rangle \setminus \{1\}) = \{0\}$, so $\chi(g) = 0$. Thus $\rho(g)$ must have both $+1$ and -1 as its eigenvalues, both with multiplicity 1. If we look at $\det \circ \rho$, which must be a degree 1 representation, we see that $\det(\rho(g)) = -1$. However, the simplicity of G guarantees that the trivial representation is the only degree 1 representation of G and this representation maps g to 1, a contradiction. $\square$