Let G be a finite group. Suppose that every $\chi \in \mathbf{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'. Show that each nonidentity coset of G' is a conjugacy class.

Proof. We note that by inflation, as G/G' is abelian, each conjugacy class of G/G' corresponds to one coset of G' and therefore two cosets cannot be in the same conjugacy class.

Note that g, g' are in gG' iff $g' = gh$ for some $h \in G'$. Let χ be an irreducible linear character of G. Then if $[a, b]$ is a generator of G', as χ is linear,

$$\chi([a,b]) = \chi(a^{-1})\chi(b^{-1})\chi(a)\chi(b) = \overline{\chi(a)}\chi(a)\overline{\chi(b)}\chi(b) = 1.$$

Thus for $h \in G'$, $\chi(h) = 1$ and $\chi(gh) = \chi(g)\chi(h) = \chi(g)$.

By assumption, for a nonlinear irreducible character χ, if $g \notin G'$, $\chi(g) = 0 = \chi(gh)$ for $h \in G'$ as $gh \notin G'$.

Therefore if $g \notin G'$ and $gh \in gG'$ then gh has the same character as g for every irreducible character χ of G. As the columns of the character table are linearly independent, g and gh must be representatives of the same column. Hence gh is in the same conjugacy class as g. That is to say every non-identity coset gG' is a conjugacy class. $\square$