Problem 2. Show that if $\chi \in Irr(G)$ and $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$.

Solution Let G be a group and let $\chi \in Irr(G)$. Suppose $\chi(1) > 1$. Then by the row orthogonality theorem,

$$1 = \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \chi(g) = \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2$$

Then by applying the arithmetic-geometric mean inequality (which states that the geometric mean of a collection of numbers is at most the arithmetic mean,) we see

$$(\prod_{g \in G} |\chi(g)|^2)^{\frac{1}{|G|}} \leq \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 = 1$$

The arithmetic-geometric mean inequality also specifies that equality is obtained if and only if all numbers in the given collection are equal. This cannot be the case here, since if it were, then we would have $1/|G| \sum_{g \in g} |\chi(g)|^2 = (|G||\chi(1)|^2)/|G| = |\chi(1)|^2 > 1$, which contradicts $\langle \chi, \chi \rangle = 1$. Hence, the geometric mean is strictly less than the arithmetic mean in this case. Moreover, since each term in the mean is positive, so is the overall mean, so we can write

$$0 \leq \prod_{g \in G} |\chi(g)|^2 < (\prod_{g \in G} |\chi(g)|)^{\frac{2}{|G|}} < 1$$

Now recall that if L/K is a Galois extension, the norm of an element $\alpha \in L$ is

$$\prod_{\sigma \in Gal(L/K)} \sigma(\alpha)$$

That is, the norm of $\alpha \in L$ is the product of all Galois conjugates of α. In our case, character values belong to the extension of $\mathbb{Q}$ by the $|G| - th$ roots of unity, say $\mathbb{Q}[\omega]$, which is a Galois extension. Let $\sigma \in Gal(\mathbb{Q}[\omega]/\mathbb{Q})$ and let $\chi \in Irr(G)$ be a character afforded by representation ρ. Then $\sigma \circ \rho$ is a representation of G, and must be irreducible (otherwise, we could apply σ^{-1} to obtain a reduction of ρ.) Moreover, each σ extends to an automorphism of $\mathbb{C}$, so applying σ to the character table of G simply permutes the rows of the table. As before, we have $\langle \sigma \circ \chi, \sigma \circ \chi \rangle = 1$ and eventually $0 \leq \prod_{g \in G} |(\sigma \circ \chi)(g)|^2 < 1$. Our choice of σ was arbitrary, so this is true for all elements of $Gal(\mathbb{Q}[\omega]/\mathbb{Q})$, yielding that

$$Norm_{\mathbb{Q}[\omega]/\mathbb{Q}}(\prod_{g \in G} |\chi(g)|^2) = \prod_{\sigma \in Gal(\mathbb{Q}[\omega]/\mathbb{Q})} \prod_{g \in G} |(\sigma \circ \chi)(g)|^2$$

is a rational number contained in $[0, 1)$.
The only rational numbers that are also algebraic integers are the rational integers, so the norm of this product is a rational integer. There is exactly one rational integer c that satisfies $0 \leq c < 1$, namely, $c = 0$, so the product has norm zero. Hence, some term of the product has norm zero, so we must have that for some $g \in G$, $|\chi(g)| = 0$. Hence, $\chi(g)$ is itself zero, so the result has been shown.