Group Theory - HW 2, Problem 5.2.6

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Problem. Prove that a nontrivial finitely generated group cannot equal its Frattini subgroup.

Solution. Let G be a finitely generated group. The Frattini subgroup of G is the collection of nongenerators of G, so it suffices to find an element of G which is not a nongenerator.

To do this, let $X = \{x_1, x_2, \ldots, x_n\}$ be a generating set of G of minimal cardinality. We have that X is finite, since G is finitely generated, and X is nonempty, since G is nontrivial. Then the element x_n cannot be a nongenerator, since $\{x_1, x_2, \ldots, x_{n-1}\}$ does not generate G by the assumption that X has minimal size. Hence, x_n does not belong to $\Phi(G)$, and we can conclude that $G \neq \Phi(G)$.