5.2.10 (R): The Frattini Subgroup of D_{2n} and D_∞

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We start with the finite case. Writing $D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1}\rangle$, we always have $\langle r\rangle$ a maximal subgroup of D_{2n}, so we must have $\Phi(D_{2n}) = \langle r^i\rangle$ for some $i \in \mathbb{Z}$. We also note that no maximal subgroup for D_{2n} contains only elements of the form sr^i as $|\langle sr^i\rangle| = 2$ and $\langle sr^i, sr^j\rangle = \langle sr^i, r^{i-j}\rangle$ and is therefore a proper subgroup when $i - j$ and n are not relatively prime and is all of D_{2n} otherwise. A general non-cyclic subgroup of a finite dihedral group has the form $\langle r^m, r^i s\rangle$, where $m|n$ and $0 \le i \le m$. A subgroup of this form is maximal if and only if $m = n/p$ where p is some prime dividing n. Since no element of $\langle r^k\rangle$ can be written as $r^i s$, we have that the Frattini subgroup of D_{2n} is,

$$\Phi(D_{2n}) = \left(\bigcap_{p|n}\langle r^p, s\rangle\right) \cap \langle r\rangle = \langle r^i\rangle$$

where i is the product of all primes dividing n.

We now consider the infinite dihedral group. The maximal subgroups of D_∞ include $\langle r\rangle$ and $\langle r^p, s\rangle$, where p is any prime. Clearly the intersection these is trivial, so we have $\Phi(D_\infty) = 1$.