Group Theory - HW 2, Problem 5.2.1

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Problem. Prove that if a nilpotent group has an element of prime order p, then so does its center.

Solution. Let G be a nilpotent group with an element g of prime order p. Let N be the normal subgroup generated by g; then, by Proposition 5.2.1, we have $N \cap Z(G) \neq 1$. Choose an element $a \in (N \cap Z(G)) \setminus \{1\}$. As N is generated by conjugates of g, we can write $a = a_1 a_2 \dots a_k$, where each a_i is a conjugate $h_i^{-1} g h_i$. We seek to show that a has order p.

First, we note that each a_i satisfies $a_i^p = 1$, since

$$a_i^p = (h_i^{-1} g h_i)^p = h_i^{-1} g^p h_i = h_i^{-1}(1) h_i = 1.$$

Next, we show that whenever a product $x_1 x_2 \cdots x_n$ belongs to the center of G, we can deduce $x_1 x_2 \cdots x_n = x_n x_1 x_2 \cdots x_{n-1}$. To see this, we note that the claim holds for a product with two elements, since $x_1 x_2 \in Z(G)$ implies $x_2^{-1} x_1 x_2 = x_1 x_2 x_2^{-1} = x_1$, and thus $x_1 x_2 = x_2 x_1$. Then we can apply the 2-element result to the n-element product as follows:

$$x_1 x_2 \cdots x_n = (x_1 x_2 \cdots x_{n-1}) x_n = x_n (x_1 x_2 \cdots x_{n-1}) = x_n x_1 x_2 \cdots x_{n-1}.$$

Now, using the above facts, we can show that $a^p = 1$ via the following computation:

$$\begin{aligned}
a^p &= (a_1 a_2 \cdots a_k)^p \\
&= (a_1 a_2 \cdots a_k)(a_1 a_2 \cdots a_k)(a_1 a_2 \cdots a_k) \cdots (a_1 a_2 \cdots a_k) \\
&= (a_1 a_2 \cdots a_k)(a_k a_1 \cdots a_{k-1})(a_1 a_2 \cdots a_k) \cdots (a_1 a_2 \cdots a_k) && \text{(permuting the 2nd factor)} \\
&= (a_1 a_2 \cdots a_k^2)(a_1 a_2 \cdots a_{k-1})(a_1 a_2 \cdots a_k) \cdots (a_1 a_2 \cdots a_k) && \text{(collecting powers of } a_k\text{)} \\
&= (a_1 a_2 \cdots a_k^2)(a_1 a_2 \cdots a_{k-1})(a_k a_1 \cdots a_{k-1}) \cdots (a_1 a_2 \cdots a_k) && \text{(permuting the 3rd factor)} \\
&= (a_1 a_2 \cdots a_k^2)(a_1 a_2 \cdots a_k)(a_1 a_2 \cdots a_{k-1}) \cdots (a_1 a_2 \cdots a_k) && \text{(absorbing } a_k \text{ into the 2nd factor)} \\
&= (a_1 a_2 \cdots a_k^2)(a_k a_1 \cdots a_{k-1})(a_1 a_2 \cdots a_{k-1}) \cdots (a_1 a_2 \cdots a_k) && \text{(permuting the 2nd factor)} \\
&= (a_1 a_2 \cdots a_k^3)(a_1 a_2 \cdots a_{k-1})(a_1 a_2 \cdots a_{k-1}) \cdots (a_1 a_2 \cdots a_k) && \text{(collecting powers of } a_k\text{)} \\
&\;\;\vdots \\
&= (a_1 a_2 \cdots a_k^p)(a_1 a_2 \cdots a_{k-1})(a_1 a_2 \cdots a_{k-1}) \cdots (a_1 a_2 \cdots a_{k-1}) \\
&= (a_1 a_2 \cdots a_{k-1})^p && \text{(using } a_k^p = 1\text{)}
\end{aligned}$$

By repeating the process found in the above steps – that is, by cyclically permuting each copy of the product $a_1 a_2 \cdots a_{k-1}$ in order to collect powers of a_{k-1} – we can show that this product is equal to $(a_1 a_2 \cdots a_{k-2})^p$. Eventually, this process yields $a^p = 1$, showing that the order of a divides p. By assumption, $a \neq 1$, so we conclude that a has order p. Since $a \in Z(G)$, this completes the proof.