Lemma. *The dihedral group of order $2p$, D_p, has exactly p non-normal subgroups when p is an odd prime.*

Proof. We note that every proper subgroup of D_p must have either 2 or p elements (by Lagrange's Theorem), and so every proper subgroup is cyclic and simple. From this, we can deduce that any non-identity element $x \in D_p$ appears in exactly one proper subgroup (the group generated by x, which must have order 2 or p). Let D_p be presented by

$$D_p = \langle r, s \mid s^2 = r^p = 1,\ s^{-1}rs = r^{-1} \rangle.$$

The elements of D_p can all be written uniquely in the form $r^n s^m$ where $n \in \{0, 1, \ldots, p-1\}$ and $m \in \{0, 1\}$.

We note that $\langle r \rangle$ is normal in G (every element commutes with r and $sr^n s = r^{-n}$, so $\langle r \rangle$ is fixed under conjugation by all generators, and hence is fixed by the entire group). Further, each of the $p-1$ distinct 2 element subgroups of the form $\langle r^n s \rangle$ for $n \neq 0$ is not normal in G since $sr^n ss = sr^n = r^{-n}s$ and $r^{-n} \neq r^n$ because $p - n = n$ has no integer solutions when p is odd. Further, $\langle s \rangle$ is not normal since $r^{-1}sr = r^{-2}s$ and $r^{-2} \neq 1$. Since we have listed a proper subgroup containing each of the elements of D_p, these are all of the proper sugroups of D_p (along with 1), and exactly p are non-normal. □

Lemma. *If G and H are groups with $(|G|, |H|) = 1$, then every subgroup of $G \times H$ has the form $L \times K$ with $L \leq G$ and $K \leq H$. Furthermore, $L \times K$ is normal in $G \times H$ if and only if L normal in G and K normal in H. As a consequence, if $\mathrm{Norm}(G)$ is the collection of normal subgroups of G, then under the assumptions above*

$$|\mathrm{Norm}(G \times H)| = |\mathrm{Norm}(G)| \cdot |\mathrm{Norm}(H)|.$$

Proof. Let $A \leq G \times H$ and suppose $(g, h) \in A$. Let $n = |g|$ and $m = |h|$. If ℓ is such that

$$\ell \equiv 1 \pmod{n}$$
$$\ell \equiv 0 \pmod{m}$$

then $(g, h)^\ell = (g, 1_H)$. Since $(|g|, |h|) = 1$ by Lagrange's Theorem, we have by the Chinese Remainder Theorem that such an ℓ exists, and so $(g, 1_H) \in A$. By a symmetric argument, $(1_g, h) \in A$. Thus, for any $g \in \pi_1(A)$ and $h \in \pi_2(A)$ (where the π_i are the canonical projections to the i-th factor), we have that $(g, 1_H)(1_g, h) = (g, h) \in A$. Thus $A = \pi_1(A) \times \pi_2(A)$.

Suppose now that A is normal in $G \times H$. Let $g \in G$ and consider $(\pi_1(A) \times 1)^{(g,1)} = (\pi_1(A))^g \times 1$. By normality of A, $(\pi_1(A))^g \times 1 \leq A$, and so

$$\pi_1\left((\pi_1(A))^g \times 1\right) = (\pi_1(A))^g \leq \pi_1(A).$$

Thus $\pi_1(A)$ is fixed by conjugation, and hence is normal. The symmetric argument shows $\pi_2(A)$ is normal. □

Problem 8. For which $n \geq 0$ is there a finite group with exactly n nonnormal subgroups?

Proof. Every abelian group has exactly 0 nonnormal subgroups. No group has exactly 1 nonnormal subgroup, since the conjugate of a nonnormal subgroup must be another nonnormal subgroup. For all other positive integers $n \geq 2$, there exists a finite group with exactly n nonnormal subgroups. We will now describe the construction of such a group.

Let A_p be D_p for odd primes p and M_{16} if $p = 2$ (we note that M_{16} has precisely 2 non-normal subgroups). The cyclic group $\mathbb{Z}_{q^{k-1}}$ has exactly k subgroups, and all of them are normal. Pick a prime q such that $q \nmid n$. Then for any prime p dividing n, we know that the orders of A_p and $\mathbb{Z}_{q^{k-1}}$ are relatively prime. By the lemma above, the nonnormal subgroups of the product $A_p \times \mathbb{Z}_{q^{k-1}}$ will be those formed by a nonnormal subgroup of A_p and any subgroup of $\mathbb{Z}_{q^{k-1}}$. It is clear that there are pk such nonnormal subgroups in the product. In particular if we choose $k = n/p$, this product will have n nonnormal subgroups. □