Problem 7. Let G be a 2-step nilpotent p-group.

(a) Show that if p is odd, then G has an abelian group word, $w(x,y)$. That is, if $x \oplus y := w(x,y)$, then $\langle G; x \oplus y \rangle$ is an abelian group.

(b) Is the same assertion true if p is even?

(c) Is the assertion true of G is a 3-step nilpotent p-group?

Proof.

(a) Because G is a 2-step nilpotent group, we know that the word w must have the form

$$w(x,y) = x^n y^m [x,y]^\ell$$

for some $n, m, \ell \in \mathbb{Z}$. Suppose that $\mathbb{G} = (G; \oplus)$ is an abelian group and let e be the identity of $(G; \cdot)$. Then we have that

$$e \oplus e = w(e,e) = e^n e^m [e,e]^\ell = e$$

so that $e \oplus e = e$ in $\mathbb{G}$. Since the only element in a group that squares to itself is the identity, we have that e must be the identity of $\mathbb{G}$ as well.

We thus have that

$$e \oplus y = y^m = y \text{ and } x \oplus e = x^n = x$$

for all $x, y \in G$ and where exponentiation is occurring in the structure $(G, \cdot)$. We may therefore assume that $n = m = 1$ and so our word is $w(x,y) = xy[x,y]^\ell$.

If $\mathbb{G}$ is abelian, we must have that $x \oplus y = y \oplus x$, so

$$xy[x,y]^\ell = yx[y,x]^\ell = yx[x,y]^{-\ell}.$$

Multiplying on the left by $x^{-1}y^{-1}$ and on the right by $[x,y]^\ell$ yields the condition

$$x^{-1}y^{-1}xy[x,y]^{2\ell} = [x,y]^{2\ell+1} = e.$$

One possible restriction on ℓ that satisfies the above condition is $2\ell + 1 = |G|$. Notice that since p is odd, ℓ is an integer. We now show that the proposed word $w(x,y) = xy[x,y]^\ell$ with $\ell = (|G|-1)/2$ does make $\langle G; x \oplus y \rangle$ an abelian group.

(Associativity) By direct computation,

$$\begin{aligned}
(x \oplus y) \oplus z &= (xy[x,y]^\ell) \oplus z \\
&= xy[x,y]^\ell z \big[xy[x,y]^\ell, z\big]^\ell \\
&= xyz[x,y]^\ell \big([xy,z]^{[x,y]^\ell} \big[[x,y]^\ell, z\big]\big)^\ell \\
&= xyz[x,y]^\ell \big([xy,z]^{[x,y]^\ell}\big)^\ell \\
&= xyz[x,y]^\ell \big([xy,z]\big)^\ell \\
&= xyz[x,y]^\ell \big([x,z]^y [y,z]\big)^\ell \\
&= xyz[x,y]^\ell [x,z]^\ell [y,z]^\ell.
\end{aligned}$$

Similarly,

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus \big(yz[y,z]^\ell\big) \\ &= xyz[y,z]^\ell \big[x, yz[y,z]^\ell\big]^\ell \\ &= xyz[y,z]^\ell \big([x,yz]\big[x,[y,z]^\ell\big]^{[y,z]^\ell}\big)^\ell \\ &= xyz[y,z]^\ell \big([x,yz]\big)^\ell \\ &= xyz[y,z]^\ell \big([x,z][x,y]^z\big)^\ell \\ &= xyz[y,z]^\ell [x,z]^\ell [x,y]^\ell. \end{aligned}$$

So $x \oplus y$ is indeed associative and $\langle G; x \oplus y\rangle$ is an abelian group as desired.

(b) Because G is 2-step nilpotent, there is some pair $x, y \in G$ such that $[x,y] = c \neq e$. By Lagrange's theorem, we know that $|c|$ divides $|G|$, so $|c|$ is a power of 2. However, we also have that $c^{2\ell+1} = e$, so $|c|$ divides $2\ell + 1$ and cannot contain a factor of 2. This is a contradiction, so $\mathbb{G}$ cannot be an abelian group.

(c) We first gather some constraints that are necessary for w to be an abelian group word on a 3-step nilpotent group G. Using commutator collection, we know that w must have the form

$$w(x,y) = x^n y^m [x,y]^\ell [x,y,x]^k [x,y,y]^j.$$

By the same argument as before, we see that $e \oplus e = e$ and so e is the identity of $\mathbb{G}$, and further that $e \oplus y = y$ and $x \oplus e = x$ implies $n = m = 1$.

In order for $\mathbb{G}$ to commute, we must have that

$$xy[x,y]^\ell [x,y,x]^k [x,y,y]^j = yx[y,x]^\ell [y,x,y]^k [y,x,x]^j$$

for all $x, y \in G$. Moving the right-hand side of the equation to the left-hand side in the usual way yields

$$[y,x,x]^{-j}[y,x,y]^{-k}[x,y]^\ell x^{-1}y^{-1}xy[x,y]^\ell [x,y,x]^k [x,y,y]^j = e.$$

Simplifying the weight 2 commutators in the middle of the left-hand expression gives

$$[y,x,x]^{-j}[y,x,y]^{-k}[x,y]^{2\ell+1}[x,y,x]^k [x,y,y]^j = e. \tag{1}$$

Furthermore, we note that

$$\begin{aligned} [y,x,z] &= [[y,x],z] \\ &= [[x,y]^{-1},z] \\ &= [x,y,z]^{-[y,x]} \\ &= [x,y,z]^{-1} \end{aligned}$$

where the last equality follows from the fact that $[x,y,z] \in Z(G)$ when G is 3-step nilpotent. Using this property on equation (1) results in the equation

$$[x,y,x]^j [x,y,y]^k [x,y]^{2\ell+1}[x,y,x]^k [x,y,y]^j = e.$$

Now, since all weight 3 commutators commute with every element of G, this further simplifies to

$$[x,y]^{2\ell+1}\left([x,y,x][x,y,y]\right)^{k+j} = e. \tag{2}$$

At this point, we have enough information to rule out the possibility of the existence of an abelian group word in a 3-nilpotent 2-group. Let $c \in [G,G] \setminus [G,G,G]$ (which is nonempty by 3-step nilpotence of G).

Let $x, y \in G$ be such that $[x, y] = c$. By Lagrange, the order of c is a power of 2, and so $(2\ell + 1, |c|) = 1$, so $\langle c \rangle = \left\langle c^{2\ell+1} \right\rangle$. In particular, $c \in \left\langle c^{2\ell+1} \right\rangle$. By equation (2),

$$\begin{aligned}
\left(c^{2\ell+1}\right)^{-1} &= ([x, y, x][x, y, y])^{k+j} \\
&\Rightarrow \left(c^{2\ell+1}\right)^{-1} \in [G, G, G] \\
&\Rightarrow c^{2\ell+1} \in [G, G, G] \\
&\Rightarrow c \in [G, G, G].
\end{aligned}$$

This contradicts our choice of c, so G cannot have an abelian group word.

$\square$