Problem 6

Any ring has a Frattini subring (= the intersection of maximal subrings), and also a Frattini ideal (= the largest $I \lhd R$ such that $I \subseteq M$ for every maximal subring $M \leq R$).
(a) Show that these notions can be different. (Produce an example where the Frattini subring is not an ideal.)
(b) State and prove the nongenerator properties for the Frattini subring of R and for the Frattini ideal of R.

Proof. (a) Any nontrivial ring has distinct Frattini ideal and subring. First observe that any subring of a ring contains 1, so the Frattini subring will as well. On the other hand, the Frattini ideal of any nontrivial ring is proper, so the Frattini ideal of any nontrivial ring must not contain 1. □

Proof. (b) Let R be a ring. For any r in R, we say R is a nongenerator if for any $X \subseteq R$, $(\langle X \cup \{r\}\rangle = R) \Rightarrow (\langle X\rangle = R)$ where $\langle X\rangle$ denotes the subring generated by X. Let F_S be the set of all nongenerators of R. Then for any $r \in R$,

$$\begin{aligned} r \notin F_S &\Leftrightarrow \exists X \subseteq R \text{ s.t. } \langle X \cup \{r\}\rangle = R \text{ and } \langle X\rangle \neq R \\ &\Leftrightarrow \exists M \prec R \text{ s.t. } r \notin M \text{ (e.g., } \langle X\rangle = M.) \end{aligned}$$

Therefore, $r \in F_S$ if and only if r is contained in every maximal subring of R. Equivalently, $r \in F_S$ if and only if r is an element of $\cap_{M \prec R} M$. Hence, the Frattini subring is F_S-the set of all nongenerators of R.
Now define $F_I = \{r \in R : r \text{ is a nongenerator of } R \text{ and } \forall a \in R, ar \text{ is a}$
$\text{nongenerator of } R\}$ This subset of R is an ideal since for any $a \in R$, and any $r \in F_I$, ar is a nongenerator by the condition defining the elements of R. Similarly, $\forall b \in R$, since $b(ar) = (ba)r$ and $(ba)r$ is a nongenerator by the definition of F_I, $b(ar)$ is a nongenerator, and so $ar \in F_I$. Moreover, since this is a subset of the set of nongenerators of R, it is an ideal which is contained in F_S. By the maximality of the Frattini ideal, we have F_I is the subset of the Frattini ideal. On the other hand, the Frattini ideal is a subset of F_I since every element of the Frattini ideal satisfies the conditions defining the elements of F_I, so the Frattini ideal is F_I.
□