Problem 6

Any ring has a Frattini subring (= the intersection of maximal subrings), and also a Frattini ideal (= the largest $I \lhd R$ such that $I \subseteq M$ for every maximal subring $M \leq R$).
(a) Show that these notions can be different. (Produce an example where the Frattini subring is not an ideal.)
(b) State and prove the nongenerator properties for the Frattini subring of R and for the Frattini ideal of R.

Proof. (a) Any nontrivial ring has distinct Frattini ideal and subring. First observe that any subring of a ring contains 1, so the Frattini subring will as well. On the other hand, the Frattini ideal of any nontrivial ring is proper, so the Frattini ideal of any nontrivial ring must not contain 1. □

Proof. (b) Let R be a ring. For any r in R, we say r is a nongenerator if for any $X \subseteq R$, $(\langle X \cup \{r\}\rangle = R) \Rightarrow (\langle X\rangle = R)$ where $\langle X\rangle$ denotes the subring generated by X. Let F_S be the set of all nongenerators of R. Then for any $r \in R$,

$$\begin{aligned} r \notin F_S &\Leftrightarrow \exists X \subseteq R \text{ s.t. } \langle X \cup \{r\}\rangle = R \text{ and } \langle X\rangle \neq R \\ &\Leftrightarrow \exists M \prec R \text{ s.t. } r \notin M \text{ (e.g., } \langle X\rangle = M.) \end{aligned}$$

Therefore, $r \in F_S$ if and only if r is contained in every maximal subring of R. Equivalently, $r \in F_S$ if and only if r is an element of $\cap_{M \prec R} M$. Hence, the Frattini subring is F_S-the set of all nongenerators of R.

Now define $F_I = \{r \in R : RrR \subseteq F_S\}$ and let $r \in F_I$, $a \in R$. Then $ar \in R(ar)R = (Ra)rR$, which is a subset of RrR. Since $r \in R$, $RrR \subseteq F_S$, so $R(ar)R \subseteq F_S$ yielding $ar \in F_I$. Similarly, $ra \in R(ra)R = Rr(aR) \subseteq RrR \subseteq F_s$, so $ra \in F_I$. Moreover, for any $r_1, r_2 \in F_I$, $R(r_1 + r_2)R = \{a(r_1 + r_2)b : a, b \in R\} = \{ar_1b + ar_2b : a, b \in R\}$. Since $r_1, r_2 \in F_I$, for any $a, b \in R$, both ar_1b and ar_2b are contained in F_S. F_S is a subring of R, so $ar_1b + ar_2b \in F_S$, and so $R(r_1 + r_2)R \subseteq F_S$. Therefore, $r_1 + r_2 \in F_I$, and so F_I is an ideal of R.

Observe that since the Frattini ideal is the largest ideal contained in all maximal subrings, it is equivalently the largest ideal contained in the Frattini subring (the intersection of all maximal subrings). Suppose $r \notin F_S$. Then $\{1\}r\{1\}$ is not a subset of F_S, and so RrR is not a subset of F_S. Therefore, $r \notin F_I$, meaning that $F_I \subseteq F_S$. In other words, F_I is an ideal contained in F_S, and so by maximality it is contained in the Frattini ideal. On the other hand, every element of the Frattini ideal satisfies the conditions that define F_I, so the Frattini ideal is a subset of F_I. Hence, the Frattini ideal and F_I are identical.

To put the definition of F_I in terms of nongenerators, $r \in F_I$ if and only if for any $a, b \in R$, $arb \in F_S$. That is to say that F_I is the set of nongenerators of R that cannot be moved from F_S via multiplication.

□