Group Theory Assignment 2

Yu Wang, Connor Meredith

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Problem 4

Lemma 1. *If φ is a retraction of a finite group G, then $ker(\varphi) \cap im(\varphi) = \{e\}$, where e is the identity of G.*

Proof. Choose any element $g \in ker(\varphi) \cap im(\varphi)$. Since $g \in im(\varphi)$, there exist an element $x \in G$ such that $g = \varphi(x)$. On the other hand, since $g \in ker(\varphi)$, then we have $e = \varphi(g) = \varphi(\varphi(x)) = \varphi(x) = g$, thus $g = e$.
Hence, $ker(\varphi) \cap im(\varphi) = \{e\}$. $\square$

Proposition. *If φ is a proper retraction of a finite group G, then $ker(\varphi) \nsubseteq \Phi(G)$*

Proof. Show it by contradiction.
Assume that φ is a proper retraction of G and $ker(\varphi) \subset \Phi(G)$. First of all, the product of $ker(\varphi)$ and $im(\varphi)$ is a subgroup of G since $ker(\varphi)$ is a normal subgroup of G and $im(\varphi)$ is a subgroup of G. We claim that $G = ker(\varphi)im(\varphi)$ because for every element $x \in G$, $x = (x\varphi(x)^{-1})\varphi(x)$. Let $\beta = x\varphi(x)^{-1}$ and $\gamma = \varphi(x)$. Obviously, $\gamma \in im(\varphi)$ and since φ is endomorphism satisfying $\varphi^2 = \varphi$, then $\varphi(\beta) = \varphi(x\varphi(x)^{-1}) = \varphi(x)\varphi(\varphi(x)^{-1}) = \varphi(x)[\varphi(\varphi(x))]^{-1} = \varphi(x)\varphi(x)^{-1} = e$ where e is the identity of G. Thus it follows that $\beta \in ker(\varphi)$. Hence, $G = ker(\varphi)im(\varphi)$. It actually means $G = \langle ker(\varphi), im(\varphi)\rangle$.

On the other hand, by the assumption that $\ker(\varphi) \subset \Phi(G)$, thus we have $G = \langle \Phi(G), \mathrm{im}(\varphi)\rangle$. Since G is a finite group, so is $\Phi(G)$, and $\Phi(G)$ is the set of all non-generators of G. By deleting elements of $\Phi(G)$ one by one, after finitely many procedures, we have $G = \langle\ \mathrm{im}(\varphi)\rangle$, which means $G = \mathrm{im}(\varphi)$.
Therefore, by the lemma 1, $ker(\varphi) = \{e\}$, which means φ is a bijection from G to itself, so φ is an automorphism of G. Then $\varphi = \varphi^2 \circ \varphi^{-1} = \varphi \circ \varphi^{-1} = 1_G$ where 1_G stands for the identity map from G to itself. It contradicts the properness of φ. $\square$