

HW #2, problem 3: Explain why, if G is a finite solvable group satisfying $\text{Fit}(G) \cong Q_8$, it must be that $8 \leq |G| \leq 48$.

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It is obvious that no group of size less than 8 can have Q_8 as a fitting subgroup. The lower bound of $|G| = 8$ is attained with $G = Q_8$, which is its own fitting group. As G is finite and solvable, $C = C_G(Q_8) = Z(Q_8) = \langle -1 \rangle$ which is of order 2. But $G/C \subseteq \text{Aut}(Q_8) \cong S_4$. So G/C is of order at most 24. Therefore $|G| \leq 48$.