HW #2, problem 1: Find all finite solvable groups which have the property that $Fit(G) \cong \mathbb{Z}_5$.

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We first show that any finite solvable group G with $Fit(G) \cong \mathbb{Z}_5$ must have order less than or equal to 20. We will denote $Fit(G) =: F$. Since G is solvable, we have $Z(F) = C_G(F) = \mathbb{Z}_5$. If p generates $C_G(F)$ then $|C_p| \leq 4$ as there are exactly 4 generators of $\mathbb{Z}_5$. Therefore $[G : \mathbb{Z}_5] = [G : \langle p \rangle] = |C_p| \leq 4$ and $|G| \leq 20$

As every abelian group is nilpotent, the only abelian group with fitting group $\mathbb{Z}_5$ is $\mathbb{Z}_5$. Using a table of small groups from page 168 of Dummit and Foote, we obtain that the non-abelian groups of order less than 20, such that 5 divides the order, are $D_{2\times5}$, $H := \langle a, b, c \mid a^5 = b^2 = c^2 = abc \rangle$, $K := \langle a, b \mid a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$, and $D_{2\times10}$. Using the GAP software, we verify that these groups are solvable and compute their Fitting subgroups:

Group	Fit(G)
$D_{2\times5}$	$\mathbb{Z}_5$
H	$\mathbb{Z}_{10}$
K	$\mathbb{Z}_5$
$D_{2\times5}$	$\mathbb{Z}_{10}$

This leaves $\mathbb{Z}_5$, $D_{2\times5}$, and K as the finite solvable groups with $\mathbb{Z}_5$ as the fitting subgroup.