

Unitriangular matrices form a Sylow p -subgroup

Assignment 1

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Prove that $U(n, p)$ is a Sylow p -subgroup of $GL(n, p)$. Deduce that every finite p -group is isomorphic with a subgroup of some $U(n, p)$.

Proof.

□

It is known that the size of $GL(n, p)$ is given by $\prod_{j=0}^{n-1} (p^n - p^j)$. This can be rewritten as $p^{\frac{n(n-1)}{2}} \prod_{j=1}^n (p^j - 1)$. Furthermore, in Robinson p. 128, it is shown that $U(n, p)$ has order $p^{\frac{n(n-1)}{2}}$.

Since none of the factors $(p^j - 1)$ of $|GL(n, p)|$ are divisible by p , we therefore have that $U(n, p)$ is a Sylow p -subgroup of $GL(n, p)$.

Now, by Cayley's Representation Theorem, every finite group (in particular, every p -group) is isomorphic to a subgroup of some S_n . We now show that S_n can be embedded into $GL(n, p)$, and hence every p -group is isomorphic to a subgroup of some $GL(n, p)$. To do this, we note that S_n is generated by a transposition $(1\ 2)$ and a cycle $(1\ 2\ \dots\ n)$. Hence, we can define an embedding on the generators of S_n by

$$(1\ 2) \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} =: A$$

and

$$(1\ 2\ \dots\ n) \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} =: B$$

These matrices have the effect of permuting the entries of vectors exactly as the elements of the

symmetric group permute the indices of those entries. That is to say, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, and

$B \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$. Hence, we have described an embedding of S_n into $GL(n, p)$. It follows that

every p -group is isomorphic to a subgroup of $GL(n, p)$, and by the Sylow Theorems, every p -group is isomorphic to a subgroup of some Sylow p -subgroup of $GL(n, p)$, which in turn must be isomorphic to $U(n, p)$, as every Sylow p -subgroup of $GL(n, p)$ is the image of $U(n, p)$ under conjugation by some group element. This completes the proof.