The decomposition of nilpotent non-abelian groups

Assignment 1

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Suppose that G is a non-abelian nilpotent group and fix $g \in G$. Then the nilpotent class of $\widetilde{G} := \langle g, G' \rangle = \langle g, [G,G] \rangle$ is smaller than that of G. Further, G can be expressed as a product of normal subgroups of smaller class.

Proof. We begin by showing that $\gamma_2 \widetilde{G} \subseteq \gamma_3 G$. This is because ${}^{\gamma_2 G}\!/_{\gamma_3 G}$ lies in the center of ${}^{G}\!/_{\gamma_3 G}$ according to the remarks on p.125 of Robinson, and so

$$\widetilde{G}/\gamma_3 G = \left(\langle g \rangle / \gamma_3 G \right) \vee \left(G' / \gamma_3 G \right) = \left(\langle g \rangle / \gamma_3 G \right) \vee \left(\gamma_2 G / \gamma_3 G \right) \leq \left(\langle g \rangle / \gamma_3 G \right) \vee Z \left(G / \gamma_3 G \right).$$

This implies that $\widetilde{G}/\gamma_3 G$ is commutative, therefore

$$\gamma_2 \widetilde{G}/\gamma_3 G = [\widetilde{G}, \widetilde{G}]/\gamma_3 G = \left[\widetilde{G}/\gamma_3 G, \widetilde{G}/\gamma_3 G \right] = 1.$$

As $\gamma_2 \widetilde{G}/\gamma_3 G = 1$, it must be the case that $\gamma_2 \widetilde{G} \subseteq \gamma_3 G$.

Now that we have shown $\gamma_2 \widetilde{G} \subseteq \gamma_3 G$, we can write $\gamma_3 \widetilde{G} = [\gamma_2 \widetilde{G}, \widetilde{G}] \subseteq [\gamma_3 G, G] = \gamma_4 G$. Continuing this process inductively we get $\gamma_i \widetilde{G} \subseteq \gamma_{i+1} G$, so that the nilpotent class of $\widetilde{G}$ is at least one less than that of G.

To show that G can be expressed as a product of normal subgroups of smaller nilpotent class, we note that $\widetilde{G}$ is normal, since $[\widetilde{G}, G] \leq [G, G] = G' \leq \widetilde{G}$. In fact, all subgroups of the form $\langle h, G' \rangle$ are normal for the same reason. Hence, G can be written as $\bigvee_{g \in G} \langle g, G' \rangle = \prod_{g \in G} \langle g, G' \rangle$, and thus is a product of normal subgroups of smaller nilpotent class. $\square$