Group Theory Assignment 1

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Problem 8

Let G be a finite nonsolvable group and N be minimal among the normal nonsolvable subgroups of G.

Part (a)

Proposition. *G has a normal subgroup N_* that is the largest normal subgroup of G properly contained in N.*

Proof. First of all, since G is a finite group, N is finite too. Thus N only has finitely many normal subgroups. Then we can assume that $N_1, N_2, \ldots, N_r$ are all of the proper normal subgroups of N. We claim that one of them is the largest proper normal subgroup as desired. Let $N_* = N_1 N_2 \ldots N_r$. By definition, N_* is a normal subgroup of N containing each $N_i (1 \leq i \leq r)$, but by the assumption, $N_1, \ldots, N_r$ are all possible proper normal subgroups of N. We claim that N_* is properly contained in N. In this case there exists j with $1 \leq j \leq r$ such that $N_* = N_j$. Therefore, N_j is the largest normal subgroup of N as desired.

For the claim, we see that each N_i is solvable by the minimality of N, so N_* is the product of finitely many solvable normal subgroups of N. Because the product of any two solvable normal subgroups is normal, it follows inductively that N_* is solvable as well, so $N_* \neq N$ as claimed. $\square$

Part (b)

Proposition. *Every normal subgroup $K \triangleleft G$ satisfies either*

(i) $N \subseteq K$ *or*

(ii) $K \subseteq (N_* : N)$

but cannot satisfy both.

Proof. Choose an arbitrary normal subgroup K of G and further suppose $N \nsubseteq K$. Then $[K, N]$ is a normal subgroup of N and is a subset of $N \cap K \subsetneq N$. Thus $[K, N] \leq N_*$ by the maximality of N_* and $K \subseteq (N_* : N)$.

Secondly, if we assume there were a normal subgroup K satisfying both (i) and (ii) at the same time, thus we would have

$$N \subseteq K \subseteq (N_* : N).$$

Then we have $[N, N] \leq N_*$, which means N/N_* is abelian since $[N, N]$ is the smallest normal subgroup such that the corresponding quotient group is abelian.

Note that N_* must be solvable as $N_* < N$ and $N_* \triangleleft G$. There is a subnormal series

$$\{1\} = M_0 \leq M_1 \leq M_2 \leq \ldots \leq M_n = N_*$$

such that each factor $M_i/M_{i-1}(1 \leq i \leq n)$ is abelian. Then we can extend the subnormal series to N, which is

$$\{1\} = M_0 \leq M_1 \leq M_2 \leq \ldots \leq M_n \leq N$$

such that each factor M_i/M_{i-1} with $1 \leq i \leq n$ and $N/M_n(= N/N_*)$ are abelian. So it shows that N is solvable, a contradiction. Therefore, no normal subgroup of G satisfies both (i) and (ii). $\square$

Part (c)

Proposition. *G is solvable if there does not exist a homomorphism from* $\mathrm{Norm}(G)$ *onto the 2-element lattice.*

Proof. We are going to prove the contrapositve of the original statement. If G is a nonsolvable group, we can define a map from $\mathrm{Norm}(G)$ onto the 2-element lattice as follows:

$$f(K) = \begin{cases} 1 & if\ N \subseteq K \\ 0 & if\ K \subseteq (N_* : N) \end{cases}$$

This map is well-defined by part (b). We claim that f is a homomorphism between $\mathrm{Norm}(G)$ and 2-element lattice $(\{0, 1\}; \vee; \wedge)$. Fixing $K_1 \in \mathrm{Norm}(G)$,

1. If $N \subseteq K_1$, then $N \subseteq K_1 \vee K$ and so $f(K_1 \vee K) = 1 = 1 \vee 0$. If $N \subseteq K$ as well, $N \subseteq K_1 \wedge K$ so $f(K_1 \wedge K) = 1 = 1 \wedge 1$.

2. If $K_1 \subseteq (N_* : N)$, then $K_1 \wedge K \subseteq (N_* : N)$ and so $f(K_1 \wedge K) = 0 = 0 \wedge 1$. If $K \subseteq (N_* : N)$ as well, $K_1K = K_1 \vee K \subseteq (N_* : N)$ so $f(K_1 \vee K) = 0 = 0 \vee 0$.

Therefore f is a homomorphism of $\mathrm{Norm}(G)$ onto the 2-element lattice. $\square$