Group Theory Assignment 1

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Problem 7

Lemma 1. *If $H, K, L \triangleleft G$ then $[H, K, L] \leq [K, L, H][L, H, K]$.*

Proof. Let $h \in H, k^{-1} \in K$, and $\ell \in L$. Then, by the Hall-Witt identity, we have that

$$[h, k, \ell]^{k^{-1}}[k^{-1}, \ell^{-1}, h]^{\ell}[\ell, h^{-1}, k^{-1}]^{h} = 1.$$

Because $[k^{-1}, \ell^{-1}, h] \in [K, L, H] \trianglelefteq G$ and $[\ell, h^{-1}, k^{-1}] \in [L, H, K] \trianglelefteq G$, we have that $[k^{-1}, \ell^{-1}, h]^{\ell}[\ell, h^{-1}, k^{-1}]^{h} \in [K, L, H][L, H, K]$. But this means that

$$[h, k, \ell]^{-k^{-1}} \in [K, L, H][L, H, K] \text{ and so } [h, k, \ell] \in [K, L, H][L, H, K]$$

since $[K, L, H][L, H, K] \trianglelefteq G$ and so is closed under conjugation and inverses. $\square$

Proposition. *If G is perfect the hypercenter of G is the center of G.*

Proof. We prove by transfinite induction that $\zeta_{\alpha}(G) = \zeta_1(G) = Z(G)$ for $\alpha \neq 0$. We note that the case $\alpha = 1$ is true. Suppose the claim is true for all $\beta < \alpha$. Suppose that $\alpha = \gamma + 1$. We then have that, by Lemma 1, the submeet property of the commutator, and $[G, \zeta_{\alpha}(G)] \leq \zeta_{\gamma}(G)$,

$$\begin{aligned}
[G, G, \zeta_{\alpha}(G)] &\leq [G, \zeta_{\alpha}(G), G][\zeta_{\alpha}(G), G, G] \\
&= [[G, \zeta_{\alpha}(G)], G][[\zeta_{\alpha}(G), G], G] \\
&\leq [Z(G), G][Z(G).G] \\
&= 1,
\end{aligned}$$

so $[G, G, \zeta_{\alpha}(G)] = [[G, G], \zeta_{\alpha}(G)] = 1$. Since G is perfect, this becomes

$$[G, \zeta_{\alpha}(G)] = 1.$$

However, $Z(G) = \zeta_1(G)$ is the greatest subgroup H of G with the property $[G, H] = 1$, so $\zeta_{\alpha}(G) \leq \zeta_1(G)$. Because we always have $\zeta_1(G) \leq \zeta_{\alpha}(G)$ for $\alpha > 1$, this implies $\zeta_{\alpha}(G) = \zeta_1(G) = Z(G)$ as desired.

If α is a limit ordinal, then

$$\zeta_{\alpha}(G) = \bigcup_{\beta < \alpha} \zeta_{\beta}(G) = \bigcup_{\beta < \alpha} Z(G) = Z(G).$$

$\square$