Problem 6. Show that if G is nilpotent, the product of two elements of finite order also has finite order.

Proof. Let $a, b \in G$ of finite order and $n = \text{lcm}(|a|, |b|)$. It is sufficient to show the result holds for the subgroup of G generated by a and b, so without loss of generality, we assume that $G = \langle a, b \rangle$. If G is 1-step nilpotent, then G is abelian and $(ab)^n = a^n b^n = 1$. By way of mathematical induction, assume that if G is k-step nilpotent then $(ab)^{n^k} = 1$. Let G be $(k+1)$-step nilpotent. Then $G/\gamma_{k+1}(G)$ is k-step nilpotent and by the induction hypothesis, $(ab)^{n^k}\gamma_{k+1}(G) = \gamma_{k+1}(G)$, yielding $(ab)^{n^k} \in \gamma_{k+1}(G) \subseteq Z(G)$.

Note that every element of $\gamma_{k+1}(G)$ is generated by elements of the form $[x_i, g_i]$ with $x_i \in \gamma_k(G)$. As these $[x_i, g_i]$ are in the center of G, $[x_i, g_1][x_i, g_2] = [x_i, g_1 g_2]$ for all $g_1, g_2 \in G$. As we are taking $G = \langle a, b \rangle$, we can rewrite $[x_i, g_i]$ as $[x_i, a]^{e_i}[x, b]^{f_i}$, giving us $[x_i, g_i]^n = [x_i, a^n]^{e_i}[x_i, b^n]^{f_i} = 1$. As $(ab)^{n^k} \in \gamma_{k+1}(G)$, we have $(ab)^{n^{k+1}} = ((ab)^{n^k})^n = \prod_{i=1}^m [x_i, g_i]^n = 1$ completing the proof. $\square$