Problem 3. Suppose that $\mathcal{W}_1$ is a "stronger" set of words than $\mathcal{W}_2$ in the sense that

$$\forall G(\forall w \in \mathcal{W}_1(G \models w = 1) \to \forall w \in \mathcal{W}_2(G \models w = 1)).$$

Show that for any group H

(a) $V_{\mathcal{W}_2}(H) \subseteq V_{\mathcal{W}_1}(H)$

(b) $M_{\mathcal{W}_1}(H) \subseteq M_{\mathcal{W}_2}(H)$

Proof.

(a) Let H be a group and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be sets of words (e.g., $\mathcal{W}_1$={Words go here}) such that $\mathcal{W}_1$ is stronger than $\mathcal{W}_2$. Recall that the verbal subgroup of H associated to $\mathcal{W}_1$ is the least normal subgroup of H such that $\forall w \in \mathcal{W}_1, H/V_{\mathcal{W}_1}(H) \models w = 1$. Since $\mathcal{W}_1$ is a stronger set of words than $\mathcal{W}_2$, we also have that $\forall w \in \mathcal{W}_2, H/V_{\mathcal{W}_1}(H) \models w = 1$. Similarly, $V_{\mathcal{W}_2}(H)$ is the least normal subgroup of H satisfying this property, so $V_{\mathcal{W}_2}(H) \subseteq V_{\mathcal{W}_1}(H)$.

(b) Let H be a group and let $\mathcal{W}_1$ and $\mathcal{W}_2$ be sets of words such that $\mathcal{W}_1$ is stronger than $\mathcal{W}_2$.
Let $w \in \mathcal{W}_2$ take n inputs and let $X = \{x_1, ..., x_n\}$ be a set of n letters. Then from our previous result, $V_{\mathcal{W}_2}(F_{Grp}(X)) \subseteq V_{\mathcal{W}_1}(F_{Grp}(X))$, so in particular, $w(x_1, ..., x_n) \in V_{\mathcal{W}_1}(F_{Grp}(X))$. By the construction of verbal subgroups, there exist $w_1, ..., w_m \in \mathcal{W}_1$, a word v, and $g_1, ..., g_l \in F_{Grp}(X)$ such that $w(x_1, ..., x_n) = v(w_1(g_1, ..., g_l), ..., w_m(g_1, ..., g_l))$ (note that we allow without loss of generality for each inner word to take the same number of inputs as not all inputs need be used.) Our choice of $w \in \mathcal{W}_2$ was arbitrary, so this construction is valid for any word in $\mathcal{W}_2$. Moreover, by the universal property of free groups, we can perform this construction for any group.
Now let $h \in M_{\mathcal{W}_1}(H)$ and let $w \in \mathcal{W}_2$. We wish to show $w(x_1, ..., x_n) = w(x_1, ..., hx_i, ..., x_n) = w(x_1, ..., x_ih..., x_n)$ for any $x_1, ..., x_n \in H$. There exist $w_1, ..., w_m \in \mathcal{W}_1$, a word v, and $g_1, ..., g_l \in H$ such that $w(x_1, ..., x_n) = v(w_1(g_1, ..., g_l), ..., w_m(g_1, ..., g_l))$. Performing the same construction on $w(x_1, ..., hx_i, ..., x_n)$, we obtain the same words, but possibly different values from H, say $g'_1, ..., g'_l$. However, each g_j, g'_j is constructible from $x_1, ..., x_n$, so $g_j = g'_j$ if x_i does not appear as a factor of g_j. Assume that x_i does appear as a factor of g_j, and fix w_k. It suffices to show that $w_k(g_1, ..., g_l) = w_k(g'_1, ..., g'_l)$. For each instance of h in g'_j, we can write $g'_j = ahx_ib = ((ahx_ib)^{a^{-1}})^a = (hx_iba)^a$. Then

$$\begin{aligned} w_k(g'_1, ..., g'_j, ..., g'_l) &= (w_k(g'_1, ..., g'_j, ..., g'_l)^{a^{-1}})^a \\ &= w_k(g_1'^{a^{-1}}, ..., hx_iba, ..., g_l'^{a^{-1}})^a \\ &= w_k(g_1'^{a^{-1}}, ..., x_iba, ..., g_l'^{a^{-1}})^a \\ &= w_k(g'_1, ..., ax_ib, ..., g'_l) \end{aligned}$$

Recursively applying this process to the finite number of instances of h and each g'_j, we arrive at $w_k(g_1, ..., g_l) = w_k(g'_1, ..., g'_l)$. Then $w(x_1, ..., x_n) = w(x_1, ..., hx_i, ..., x_n)$. The proof of $w(x_1, ..., x_n) = w(x_1, ..., x_ih, ..., x_n)$ is nearly identical, and only varies at the conjugation step. Instead of conjugating by $a^{-1}a$, conjugation by bb^{-1} suffices. Then $h \in M_{\mathcal{W}_2}(H)$, and so $M_{\mathcal{W}_1}(H) \subseteq M_{\mathcal{W}_2}(H)$.

$\square$