Problem 2. Give an example of a marginal subgroup that is not fully invariant.

Proof. We know that the center of a group is the marginal subgroup for the commutator word. The center of $\mathbb{Z}/2 \times S_3$ is not fully invariant. To see this, notice that the center of a product is the product of the respective centers. Since S_3 has trivial center, $Z(\mathbb{Z}/2 \times S_3) = \mathbb{Z}/2 \times e$, where e is the identity element of S_3.

Now let $\phi : \mathbb{Z}/2 \times S_3 \to \mathbb{Z}/2 \times S_3$ be the map given by $\phi(1, x) \mapsto \big(0, (1\ 2)\big)$, i.e. the map that takes $\mathbb{Z}/2$ to the subgroup of S_3 generated by the 2-cycle $(1\ 2)$. Then the image of ϕ does not lie in the center of $\mathbb{Z}/2 \times S_3$. $\square$