Assignment 1, problem 1: The decomposition of nilpotent non-abelian groups

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> Suppose that G is a non-abelian nilpotent group and fix $g \in G$. Then the nilpotent class of $\widetilde{G} := \langle g, G' \rangle = \langle g, [G, G] \rangle$ is smaller than that of G. Further, G can be expressed as a product of normal subgroups of smaller class.

Proof. If $g \in G'$, then $\gamma_1 \widetilde{G} = \widetilde{G} = G' = \gamma_2 G$, and thus the nilpotent class of $\widetilde{G}$ is exactly 1 less than that of G and we are done.

Thus suppose $g \notin G'$. We first show that $\gamma_2 \widetilde{G} \subseteq \gamma_3 G$. Let $w \in \widetilde{G}' = \gamma_2 \widetilde{G}$. Since G' is a normal subgroup of G, we may rewrite $\langle g, G' \rangle = \langle g \rangle G' = G' \langle g \rangle$, so every $x \in \widetilde{G}$ can be written either as $g^{e_1} y_1$ or as $y_2 g^{e_2}$ for some $e_1, e_2 \in \mathbb{Z}$ and $y_1, y_2 \in G'$. We use this fact to rewrite $w = [\widetilde{g}_1, \widetilde{g}_2] = [g_1' g^s, g^t g_2']$ for some $s, t \in \mathbb{Z}$ and $g_1', g_2' \in G'$. If $s = 0$, then $w = [g_1', g^t g_2'] \in [G', G] = \gamma_3 G$. Similarly, if $t = 0$, then we have $w = [g_1' g^s, g_2'] \in [G, G'] = [G', G] = \gamma_3 G$. Hence, $w \in \gamma_3 G$ for all $w = [g_1' g^s, g^t g_2'] \in \gamma_2 \widetilde{G}$ with $\min\{|s|, |t|\} = 0$. This forms the base case of our inductive argument.

Now, for the inductive step, suppose that we have $[g_1' g^s, g^t g_2'] \in [G', G]$ whenever $\min\{|s|, |t|\} = k$. Let $w = [g_1' g^{k+1}, g^t g_2']$ for some $g_1', g_2' \in G'$ and for some t with $|t| \geq k + 1$. Then, we have the following:

$$
\begin{aligned}
w &= [g_1' g^{k+1}, g^t g_2'] \\
&= g^{-k-1} (g_1')^{-1} (g_2')^{-1} g^{-t} g_1' g^{k+1} g^t g_2' \\
&= g^{-1} g^{-k} (g_1')^{-1} (g_2')^{-1} g^{-t} g_1' g^k g^t g_2' g [g, g_2'] \\
&= g^{-1} [g_1' g^k, g^t g_2'] g [g, g_2']
\end{aligned}
$$

Now, we note that $[g, g_2'] \in [G, G'] = [G', G]$. Furthermore, by the inductive hypothesis, $[g_1' g^k, g^t g_2'] \in [G', G]$. Thus, since $[G', G] = \gamma_3 G$ is a normal subgroup of G, we also have $g^{-1} [g_1' g^k, g^t g_2'] g \in [G', G]$, and thus $w \in [G', G] = \gamma_3 G$.

An entirely similar argument would show that the element $w' := [g_1' g^{-k-1}, g^t g_2']$ is an element of $\gamma_3 G$, so we can conclude that every element of the form $[g_1' g^s, g^t g_2']$, where $|s| = k + 1$ and $|t| \geq |s|$, is an element of $\gamma_3 G$. We now let $w = [g_1' g^s, g^{k+1} g_2']$, where $|s| \geq k + 1$, and again show $w \in [G', G]$. Again, this is sufficient to show that every element of the form $[g_1' g^s, g^t g_2']$, with $|s| \geq k + 1$ and $|t| = k + 1$, belongs to $\gamma_3 G$. Hence, showing that $w \in \gamma_3 G$ will complete the proof of our claim for

$\min\{|s|,|t|\} = k+1$. We prove $w \in \gamma_3 G$ as follows:

$$\begin{aligned}
w &= g^{-s}(g_1')^{-1}(g_2')^{-1}g^{-k-1}g_1'g^s g^{k+1}g_2' \\
&= g^{-s}(g_1')^{-1}(g_2')^{-1}g^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= g^{-s}(g_1')^{-1}[g_2',g]g^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= g^{-s}[g_1',[g,g_2']]\,[g_2',g](g_1')^{-1}g^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= \Big[g^s,[[g,g_2'],g_1']\Big]\,[g_1',[g,g_2']]\,g^{-s}[g_2',g](g_1')^{-1}g^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= \Big[g^s,[[g,g_2'],g_1']\Big]\,[g_1',[g,g_2']]\,[g^s,[g,g_2']]\,[g_2',g]g^{-s}(g_1')^{-1}g^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= \Big[g^s,[[g,g_2'],g_1']\Big]\,[g_1',[g,g_2']]\,[g^s,[g,g_2']]\,[g_2',g]g^{-s}[g_1',g]g^{-1}(g_1')^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= \overbrace{\Big[g^s,[[g,g_2'],g_1']\Big]}^{\in [G,[G',G]]=\gamma_4 G}\,\overbrace{[g_1',[g,g_2']]}^{\in [G,G']}\,\overbrace{[g^s,[g,g_2']]}^{\in [G,G']}\,\overbrace{[g_2',g]}^{\in [G',G]}\,\overbrace{[g^s,[g,g_1']]}^{\in [G,G']}\,\overbrace{[g_1',g]}^{\in [G',G]}\,g^{-s-1}(g_1')^{-1}(g_2')^{-1}g^{-k}g_1'g^{s+1}g^k g_2' \\
&= (\text{an element of }\gamma_3 G)[g_1'g^{s+1},g^k g_2']
\end{aligned}$$

Note that the last equality holds because $\gamma_4 G \subseteq \gamma_3 G$, and because $[G,G'] = [G',G] = \gamma_3 G$. Now, by the induction hypothesis, we have $[g_1'g^{s+1},g^k g_2'] \in \gamma_3 G$, and thus we have $w \in \gamma_3 G$. This completes our inductive proof that $\gamma_2\widetilde{G} \subseteq \gamma_3 G$.

Now that we have shown $\gamma_2\widetilde{G} \subseteq \gamma_3 G$, we can write $\gamma_3\widetilde{G} = [\gamma_2\widetilde{G},\widetilde{G}] \subseteq [\gamma_3 G, G] = \gamma_4 G$. Continuing this process inductively we get $\gamma_i\widetilde{G} \subseteq \gamma_{i+1}G$, so that the nilpotent class of $\widetilde{G}$ is at least one less than that of G.

Finally, to show that G can be expressed as a product of normal subgroups of smaller nilpotent class, we note that $\widetilde{G}$ is normal, since for all $h \in G$,

$$h\widetilde{G} = h\langle g\rangle G' = h\langle g\rangle G'G' = (hG')(\langle g\rangle G') \overset{!}{=} (\langle g\rangle G')(hG') = (\langle g\rangle G')(G'h) = \langle g\rangle G'h = \widetilde{G}h.$$

The step marked with ! requires some justification, but ultimately follows from the normality of G':

$$\begin{aligned}
(hG')(\langle g\rangle G') &= (hG')(\bigcup_{a\in\langle g\rangle} aG') \\
&= \bigcup_{a\in\langle g\rangle} (hG')(aG') \\
&= \bigcup_{a\in\langle g\rangle} (aG')(hG') \\
&= (\bigcup_{a\in\langle g\rangle} aG')(hG') \\
&= (\langle g\rangle G')(hG').
\end{aligned}$$

Now that we have that $\widetilde{G} = \langle g \rangle G'$ is normal, it follows that G can be written as $\bigvee_{g \in G} \langle g \rangle G' = \prod_{g \in G} \langle g \rangle G'$, and hence is a product of normal subgroups of smaller nilpotent class. $\square$

Assignment 1, problem 2: Unitriangular matrices form a Sylow p-subgroup.

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Prove that $U(n,p)$ is a Sylow p-subgroup of $GL(n,p)$. Deduce that every finite p-group is isomorphic with a subgroup of some $U(n,p)$.

Proof. $\square$

It is known that the size of $GL(n,p)$ is given by $\prod_{j=0}^{n-1}(p^n - p^j)$. We now show that we can rewrite this as $p^{\frac{n(n-1)}{2}} \prod_{j=1}^{n}(p^j - 1)$. We proceed by induction. When $n = 2$, we have

$$|GL(n,p)| = (p^2-1)(p^2-p) = p^4 - p^3 - p^2 + p = p(p^3 - p^2 - p + 1) = p^{\frac{2\cdot 1}{2}}(p-1)(p^2-1)$$

so the base case has been shown. Now assume $|GL(k,p)| = p^{\frac{k(k-1)}{2}} \prod_{j=1}^{k}(p^k - 1)$. Then

$$\begin{aligned}
|GL(k+1,p)| &= \prod_{j=0}^{k}(p^{k+1} - p^j) \\
&= (p^{k+1} - 1) \cdot p^k \prod_{j=1}^{k-1}(p^k - p^j) \\
&= (p^{k+1} - 1) \cdot p^{\frac{k(k+1)}{2}} \prod_{j=1}^{k}(p^j - 1) \\
&= p^{\frac{k(k+1)}{2}} \prod_{j=1}^{k+1}(p^j - 1)
\end{aligned}$$

Hence, we have $|GL(n,p)| = p^{\frac{n(n-1)}{2}} \prod_{j=1}^{n}(p^j - 1)$. Furthermore, in Robinson p. 128, it is shown that $U(n,p)$ has order $p^{\frac{n(n-1)}{2}}$. Since none of the factors $(p^j - 1)$ of $|GL(n,p)|$ are divisible by p, we therefore have that $U(n,p)$ is a Sylow p-subgroup of $GL(n,p)$.

Now, by Cayley's Representation Theorem, every finite group (in particular, every p-group) is isomorphic to a subgroup of some S_n. We now show that S_n can be embedded into $GL(n,p)$, and hence every p-group is isomorphic to a subgroup of some $GL(n,p)$. To do this, we note that S_n is generated by a transposition $(1\ 2)$ and a cycle $(1\ 2\ \ldots\ n)$. Hence, we can define an embedding on the generators of S_n by

$$(1\ 2) \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} =: A$$

and

$$(1\ 2\ \ldots\ n) \mapsto \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} =: B$$

These matrices have the effect of permuting the entries of vectors exactly as the elements of the symmetric group permute the indices of those entries. That is to say, $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$, and $B \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_n \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$. Hence, we have described an embedding of S_n into $GL(n, p)$. It follows that every p-group is isomorphic to a subgroup of $GL(n, p)$, and by the Sylow Theorems, every p-group is isomorphic to a subgroup of some Sylow p-subgroup of $GL(n, p)$, which in turn must be isomorphic to $U(n, p)$, as every Sylow p-subgroup of $GL(n, p)$ is the image of $U(n, p)$ under conjugation by some group element. This completes the proof.

References

For the first problem our proof was inspired by the following paper: `https://projecteuclid.org/download/pdf_1/euclid.rmjm/1250128843` which proved a result similar to what we needed. Our proof however uses a different approach then the authors'.

We also thank Daniel Martin for his help in figuring out why $\langle g \rangle G'$ is abelian.