Examples, Properties, Applications

Math 6270

G	1 1	k_2 g_2	$\cdots$ $\cdots$	k_r g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

Each d_i divides $|G| = 1^2 + d_2^2 + \cdots + d_r^2$
Each k_j divides $|G| = 1 + k_2 + \cdots + k_r$

Character tables of abelian groups

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$\mathbb{Z}_2$	$\begin{matrix}1\\0\end{matrix}$	$\begin{matrix}1\\1\end{matrix}$
ξ_1	1	1
ξ_2	1	-1

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$\mathbb{Z}_3$	1 0	1 1	1 2
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χ_2	1	ω	ω^2
χ_2^2	1	ω^2	ω

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χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_2^2	1	-1	1	-1
χ_2^3	1	$-i$	-1	i

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$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{matrix}1\\(0,0)\end{matrix}$	$\begin{matrix}1\\(0,1)\end{matrix}$	$\begin{matrix}1\\(1,0)\end{matrix}$	$\begin{matrix}1\\(1,1)\end{matrix}$
$\xi_1(x)\xi_1(y)$	1	1	1	1
$\xi_2(x)\xi_1(y)$	1	1	-1	-1
$\xi_1(x)\xi_2(y)$	1	-1	1	-1
$\xi_2(x)\xi_2(y)$	1	-1	-1	1

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χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	?	?	?	?	?

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The entries in the last row can be determined by:
(i) finding the missing irrep, (ii) column orthogonality,

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The entries in the last row can be determined by:
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χ_5	2	-2	0	0	0

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The table for either of $D_4(D_8)$ or Q_8 is:

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The groups D_4 and Q_8 can be distinguished by the fact that $\det(\chi_5^{Q_8}) = \chi_1 \neq \det(\chi_5^{D_4})$.

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S_4	1 1	3 (1 2)(3 4)	8 (1 2 3)	4 (1 2)	4 (1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

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χ_1	1	1	1	1	1
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χ_3	3	-1	0	$-\phi^{-1}$	ϕ
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

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χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

χ_2	$1+2\cos(0)$	$1+2\cos(\pi)$	$1+2\cos(2\pi/3)$	$1+2\cos(2\pi/5)$	$1+2\cos(4\pi/5)$

Silly applications

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The first column of the character table for G consists of at most p integers $d_1, \ldots, d_r$ such that (i) $d_1 = 1$, (ii) $d_j | p$ for all j, and (iii) $d_1^2 + \cdots + d_r^2 = p$.

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Same idea. The smallest number n that is a sum of squares $d_1^2 + \cdots + d_r^2$ where $d_1 = 1$, some $d_j > 1$, and all d_j divide n is 6. $\square$

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If $|G| = pq$, $q < p$ primes, and G is nonabelian, then $p \equiv 1 \pmod{q}$.

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Possible degrees are $d_j \in \{1, q\}$, and some $d_j = q$. Assume that there are m degrees satisfying $d_j = q$. The number of d_j equal to 1 is $n := [G : G'] \in \{1, q, p\}$.

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Possible degrees are $d_j \in \{1, q\}$, and some $d_j = q$. Assume that there are m degrees satisfying $d_j = q$. The number of d_j equal to 1 is $n := [G : G'] \in \{1, q, p\}$. We have $pq = d_1^2 + \cdots + d_r^2 = mq^2 + n$.

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Possible degrees are $d_j \in \{1, q\}$, and some $d_j = q$. Assume that there are m degrees satisfying $d_j = q$. The number of d_j equal to 1 is $n := [G : G'] \in \{1, q, p\}$. We have $pq = d_1^2 + \cdots + d_r^2 = mq^2 + n$. From this, $q|n$, so $q = n$.

More silly applications

Theorem

If $|G| = p^2$, p prime, then G is abelian.

Proof.

Each degree d_j must be $1, p$ or p^2. The choices $d_j = p$ or p^2 are too big if $1^2 + d_2^2 + \cdots + d_r^2 = p^2$. $\square$

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Unsilly applications

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Each summand on the left is an algebraic integer, but $-1/p$ is not. $\square$

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If H is a subgroup of G of p-power index with a nontrivial central element h, then h^G is a conjugacy class of G of p-power size. Conversely if h^G is a conjugacy class of G of p-power size, then $C_G(h)$ is a subgroup of G of p-power index with a nontrivial central element h.

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	1	15	20	12	12
A_5	1	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 2\ 3\ 5\ 4)$
χ_1	1	1	1	1	1
χ_2	3	-1	0	ϕ	ϕ^{-1}
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There are zeros at the centers of the colored crosses.

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What we are seeing here is important:

Let G be a nonabelian simple group with irreducible character ψ of degree p. If $\rho\colon G \to \mathrm{GL}_p(\mathbb{C})$ affords ψ, then ρ is faithful. In fact, $K_\psi = Z_\psi = \{1\}$.

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What we are seeing here is important: a character is vanishing on the nonidentity elements of a subgroup, $Z(P)$.

Recognizing multiples of the regular representation

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Lemma

Let $\rho\colon H \to GL_n(\mathbb{C})$ be a (not necessarily irreducible) representation of a finite group, and let ψ be the character afforded by ρ. If ψ vanishes on $H \setminus \{1\}$, then $|H|$ divides $\psi(1)$.

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Compare χ_1 and ψ:

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$\langle \chi_1, \psi \rangle = \frac{\psi(1)}{|H|}$ must be an integer. Now use that representations are determined by their characters. $\square$

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To be proved: If there is an irreducible representation $\rho\colon G \to \mathrm{GL}_p(\mathbb{C})$ (affording ψ) then a Sylow p-subgroup $P \leq G$ has order p.

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The converse is false.

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The converse is false. $\mathrm{PSL}_2(q)$ has Sylow 3-subgroups of size 3 for infinitely many prime powers q, but only $\mathrm{PSL}_2(5)$ and $\mathrm{PSL}_2(7)$ have irreps of degree 3.