Practice problems: Linear transformations of order 2

Assume that V is finite dimensional and that $T\colon V \to V$ satisfies $T^2 = I$. Here you will argue that V is a direct sum of its T-symmetric and T-antisymmetric parts.

(1) Suppose that $T^2 = I$. What are the possible minimal polynomials for T?

(2) What are the possible characteristic polynomials for T?

(3) Must T be diagonalizable?

(4) Explain why $V = V_1 \oplus V_{-1}$.

(5) You know some examples of linear transformations of order 2: $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})\colon M \mapsto M^t$, and $T'\colon \mathbb{P}_k(t) \to \mathbb{P}_k(t)\colon f(t) \mapsto f(-t)$. What does the above tell you about the spaces operated on by T and T'?