Simple groups

Modern Algebra 1

Fall 2016

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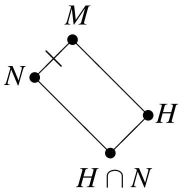

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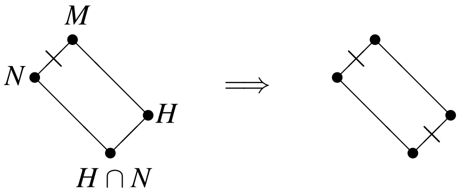

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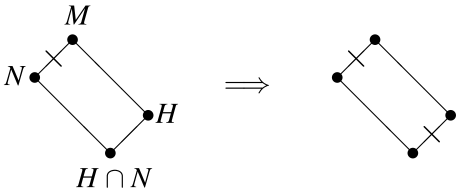

$$M/N \cong H/(H \cap N)$$

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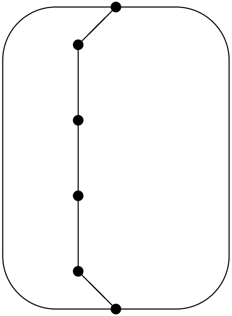

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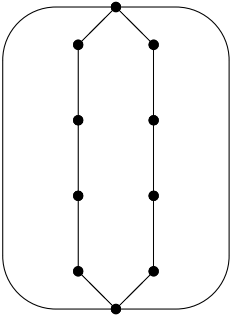

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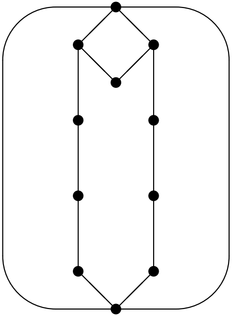

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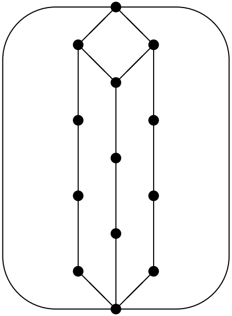

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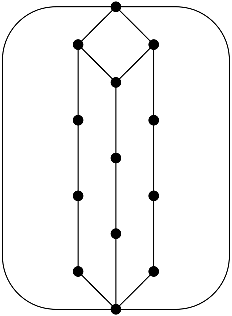

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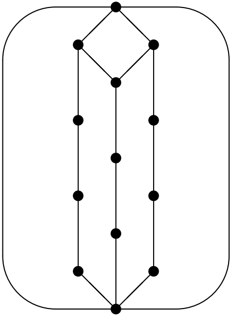

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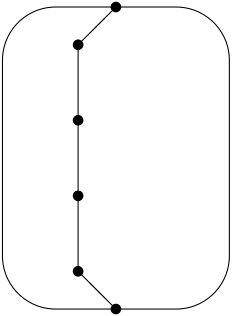

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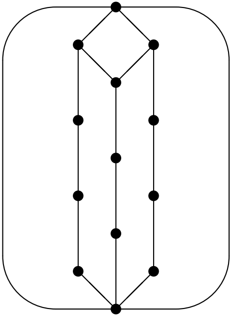

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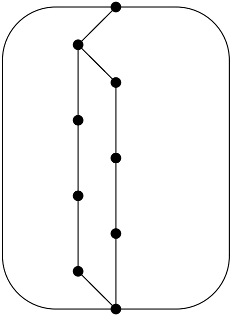

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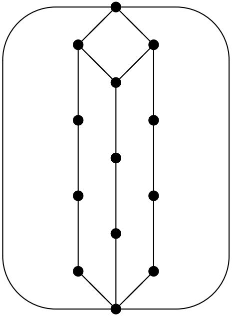

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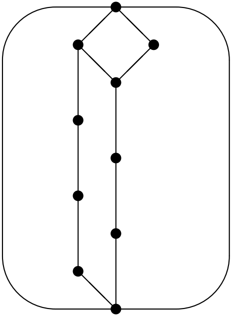

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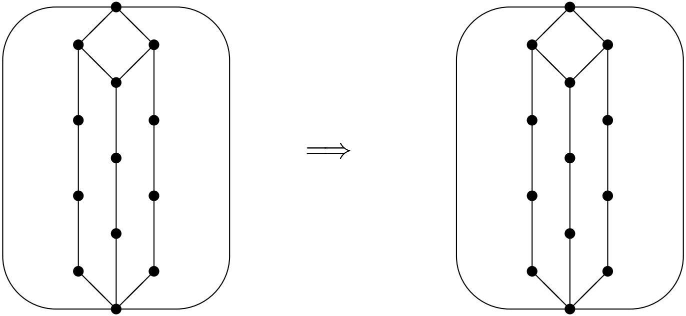
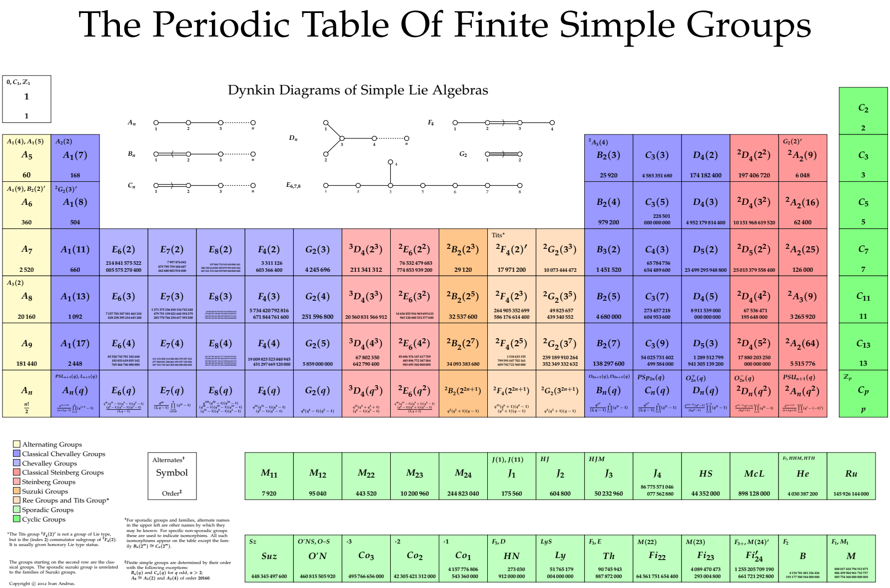


The classification of finite simple groups (CFSG)


The Periodic Table Of Finite Simple Groups
Dynkin Diagrams of Simple Lie Algebras
A_n
B_n
C_n
D_n
$E_{6,7,8}$
F_4
G_2
$0, C_1, \mathbb{Z}_1$ 1 1
A_5 60
$A_1(7)$ 168
A_6 360
$A_1(8)$ 504
A_7 2 520
$A_1(11)$ 660
$E_6(2)$
$E_7(2)$
$E_8(2)$
$F_4(2)$ 3 311 126 603 366 400
$G_2(3)$ 4 245 696
${}^3D_4(2^3)$ 211 341 312
${}^2E_6(2^2)$ 76 532 479 683 774 853 939 200
${}^2B_2(2^3)$ 29 120
Tits*
${}^2F_4(2)'$ 17 971 200
${}^2G_2(3^3)$ 10 073 444 472
A_8 20 160
$A_1(13)$ 1 092
$E_6(3)$
$E_7(3)$
$E_8(3)$
$F_4(3)$ 5 734 420 792 816 671 844 761 600
$G_2(4)$ 251 596 800
${}^3D_4(3^3)$ 20 560 831 566 912
${}^2E_6(3^2)$
${}^2B_2(2^5)$ 32 537 600
${}^2F_4(2^3)$ 264 905 352 699 586 176 614 400
${}^2G_2(3^5)$ 49 825 657 439 340 552
A_9 181 440
$A_1(17)$ 2 448
$E_6(4)$
$E_7(4)$
$E_8(4)$
$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000
$G_2(5)$ 5 859 000 000
${}^3D_4(4^3)$ 67 802 350 642 790 400
${}^2E_6(4^2)$
${}^2B_2(2^7)$ 34 093 383 680
${}^2F_4(2^5)$
${}^2G_2(3^7)$ 239 189 910 264 352 349 332 632
A_n $\frac{n!}{2}$
$A_n(q)$
$E_6(q)$
$E_7(q)$
$E_8(q)$
$F_4(q)$
$G_2(q)$
${}^3D_4(q^3)$
${}^2E_6(q^2)$
${}^2B_2(2^{2n+1})$
${}^2F_4(2^{2n+1})$
${}^2G_2(3^{2n+1})$
$B_2(3)$ 25 920
$C_3(3)$ 4 585 351 680
$D_4(2)$ 174 182 400
${}^2D_4(2^2)$ 197 406 720
${}^2A_2(9)$ 6 048
$B_2(4)$ 979 200
$C_3(5)$ 228 501 000 000 000
$D_4(3)$ 4 952 179 814 400
${}^2D_4(3^2)$ 10 151 968 619 520
${}^2A_2(16)$ 62 400
$B_3(2)$ 1 451 520
$C_4(3)$ 65 784 756 654 489 600
$D_5(2)$ 23 499 295 948 800
${}^2D_5(2^2)$ 25 015 379 558 400
${}^2A_2(25)$ 126 000
$B_2(5)$ 4 680 000
$C_3(7)$ 273 457 218 604 953 600
$D_4(5)$ 8 911 539 000 000 000 000
${}^2D_4(4^2)$ 67 536 471 195 648 000
${}^2A_3(9)$ 3 265 920
$B_2(7)$ 138 297 600
$C_3(9)$ 54 025 731 402 499 584 000
$D_5(3)$ 1 289 512 799 941 305 139 200
${}^2D_4(5^2)$ 17 880 203 250 000 000 000
${}^2A_2(64)$ 5 515 776
$B_n(q)$
$C_n(q)$
$D_n(q)$
${}^2D_n(q^2)$
${}^2A_n(q^2)$
C_2 2
C_3 3
C_5 5
C_7 7
C_{11} 11
C_{13} 13
C_p p
Alternating Groups
Classical Chevalley Groups
Chevalley Groups
Classical Steinberg Groups
Steinberg Groups
Suzuki Groups
Ree Groups and Tits Group*
Sporadic Groups
Cyclic Groups
Alternates†
Symbol
Order‡
M_{11} 7 920
M_{12} 95 040
M_{22} 443 520
M_{23} 10 200 960
M_{24} 244 823 040
J_1 175 560
J_2 604 800
J_3 50 232 960
J_4 86 775 571 046 077 562 880
HS 44 352 000
McL 898 128 000
He 4 030 387 200
Ru 145 926 144 000
Suz 448 345 497 600
$O'N$ 460 815 505 920
Co_3 495 766 656 000
Co_2 42 305 421 312 000
Co_1 4 157 776 806 543 360 000
HN 273 030 912 000 000
Ly 51 765 179 004 000 000
Th 90 745 943 887 872 000
Fi_{22} 64 561 751 654 400
Fi_{23} 4 089 470 473 293 004 800
Fi_{24}' 1 255 205 709 190 661 721 292 800
B
M
*The Tits group ${}^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of ${}^2F_4(2)$. It is usually given honorary Lie type status.
The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.
†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphisms. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.
‡Finite simple groups are determined by their order with the following exceptions: $B_n(q)$ and $C_n(q)$ for q odd, $n > 2$; $A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.
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Sketch of proof* that the center Z of $SL(V)$ is a maximal normal subgroup.

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