Practice Problems about Commutative Rings

Topics: Basic definitions and examples, lattice of subrings/ideals, quotient rings, chain conditions, rings of fractions, Chinese Remainder Theorem, Euclidean domains, PIDs, UFDs, polynomial rings, irreducibility criteria for polynomials.

(1) Let R be a commutative ring with identity.
   (a) Suppose that I is an ideal of R that is contained in the principal ideal (a). Show that there is an ideal J of R such that $I = (a)J$.

   (b) Now suppose that $R = \mathbb{C}[x, y]$. Give an example of two ideals $I \subseteq A$ of R for which there is no ideal J satisfying $I = AJ$.

(2) Show that if R is a PID and S is an integral domain containing no subfield, then any homomorphism $\varphi\colon R \to S$ is injective.

(3) Either prove or disprove the following statement, with a full justification. If R is an integral domain that is not a field, then the polynomial ring $R[x]$ can never be a principal ideal domain.

(4) Prove that the subring of $\mathbb{Q}[x]$ consisting of all polynomials with integer constant term is not a UFD.

(5) Show that x, y and z are irreducible and prime elements of $k[x, y, z]$, where k is a field. Prove that $k[x, y, z]/(xy - z^2)$ is an integral domain.

(6) Let I be a nonzero ideal in $\mathbb{Z}[i]$. Show that I is a prime ideal if and only if it is a maximal ideal.

(7) Let R be a commutative and associative ring with multiplicative identity $1 \neq 0$ and let I be an ideal of R. Suppose that I is not finitely generated and that the only ideal of R not finitely generated and containing I is I itself. Then show that I is a prime ideal. [Hint: You may want to make use of $J_a := \{r \in R : ra \in I\}$ for $a \in R$.]

(8) (a) Let $n \in \mathbb{Z}$, $n \geq 1$, and let I be the ideal generated by n and x in $\mathbb{Z}[x]$. Show that I is a maximal ideal if and only if n is prime.

(b) Show that $\mathbb{Z}[x]$ is not isomorphic, as a ring, to $\mathbb{Z}$.

(c) (Defn of group ring given.) Show that if G is any nontrivial group, the group ring $\mathbb{Z}G$ has at least four units. Deduce that $\mathbb{Z}[x]$ is not isomorphic to any group ring $\mathbb{Z}G$.

(9) Let R be a commutative ring with unity. Suppose for each $r \in R$, there exists an integer $n_r > 1$ such that $r^{n_r} = r$. Show that every prime ideal in R is maximal.

(10) (a) (i) Prove that the integral domain $\mathbb{Z}[i]$ is a Euclidean domain.

(ii) What are its units?

(iii) Give an example of a maximal ideal of $\mathbb{Z}[i]$.

(b) (i) Prove that the integral domain $\mathbb{Z}[x]$ is not a Euclidean domain.

(ii) What are its units?

(iii) Give an example of a maximal ideal of $\mathbb{Z}[x]$.

(c) (i) Prove that the integral domain $\mathbb{Z}[\sqrt{-5}]$ is not a Euclidean domain.

(ii) What are its units?

(11) (a) Show that complex conjugation restricts to an automorphism of $\mathbb{Z}[\sqrt{-5}]$.

(b) Show that ± 1 are the only units of $\mathbb{Z}[\sqrt{-5}]$.

(c) Show that $2+\sqrt{-5}$, $2-\sqrt{-5}$, and 3 are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.

(d) From the equation $3 \cdot 3 = (2+\sqrt{-5})(2-\sqrt{-5})$, show that $\mathbb{Z}[\sqrt{-5}]$ is not a principle ideal domain.

(12) Show that if R is an integral domain, $F \leq R$ is a subring that is a field, and R is finite dimensional as an F-space, then R is a field.

(13) Let R be a commutative unital ring with $1 \neq 0$. Show that if every proper principal ideal of R is a prime ideal, then R is a field.

(14) Let R be a commutative ring. Show that an ideal I in R is prime if and only if it satisfies both of the following properties:
(a) if $I = I_1 \cap I_2$ for two ideals I_1, I_2 in R, then either $I_1 = I$ or $I_2 = I$;

(b) if $a \in R$ and $a^n \in I$ for some positive integer n, then $a \in I$.

(15) Prove that if R is a domain and $a \neq 0$ is not a unit in R, then (a, x) is not a principal ideal in $R[x]$. Explain why $\mathbb{Q}[x]$ is a Euclidean domain, but $\mathbb{Q}[x, y]$ is not.

(16) Let R be a commutative ring with identity, I_1 and I_2 ideals in R, and $\varphi\colon R \to R/I_1 \times R/I_2$ the canonical mapping.
(a) Describe $\ker(\varphi)$ and show that if $I_1 + I_2 = R$ then $\ker(\varphi) = I_1 I_2$.

(b) Prove that when $I_1 + I_2 = R$ the mapping φ is surjective.

(c) Show that $(Z_{100})^\times$ is isomorphic to $(Z_4)^\times \times (Z_{25})^\times$.