Linear Algebra
MIDTERM

Name:____________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. For each of the following statements, answer True or False. Give a brief (1-sentence) justification for each answer.

(a) Every homogeneous system is consistent.

True. Any homogeneous system $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as a solution.

(b) The zero function $T(\mathbf{x}) = \mathbf{0}$ is a linear transformation.

True. The zero transformation satisfies $T(\mathbf{x}+\mathbf{y}) = \mathbf{0} = \mathbf{0}+\mathbf{0} = T(\mathbf{x})+T(\mathbf{y})$ and $T(r\mathbf{x}) = \mathbf{0} = r\mathbf{0} = rT(\mathbf{x})$.

(c) Upper triangular matrices are invertible.

False. For example, the zero matrix is upper triangular and not invertible.

(d) If A is a 3×3 matrix, then $\det(5A) = 5\det(A)$.

False. For example, if $A = I$ is the 3×3 identity matrix, then $\det(5A) = 125 \neq 5 = 5\det(A)$.

2. Use linear algebra to find a curve of the form $y = ax^2 + bx + c$ that passes through the points $(x, y) = (-1, 8), (0, 1)$ and $(1, -2)$. (Hint: start by plugging the points into the equation.)

Substituting points yields a system of linear equations in the unknowns a, b, c that has the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -2 \end{array}\right].$$

The solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, which yields $y = 2x^2 - 5x + 1$.

3. Explain why if A is invertible, then A^TA is invertible.

If A is invertible (say with inverse B), then A^T is also invertible (with inverse B^T). The product A^TA of two invertible matrices is again invertible (with inverse BB^T).

4. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are column vectors. How are the determinants of the 3×3 matrices $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ and $[(\mathbf{u}+\mathbf{v})\ \ (\mathbf{v}+\mathbf{w})\ \ (\mathbf{w}+\mathbf{u})]$ related? (Hint: the determinant is the unique function of the columns that has a certain three properties.)

Using the multilinearity and alternating properties of the determinant we find that

$$\begin{aligned} [(\mathbf{u}+\mathbf{v})\ \ (\mathbf{v}+\mathbf{w})\ \ (\mathbf{w}+\mathbf{u})] &= [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{u}] + [\mathbf{u}\ \ \mathbf{w}\ \ \mathbf{w}] + [\mathbf{u}\ \ \mathbf{w}\ \ \mathbf{u}] \\ &\quad + [\mathbf{v}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{v}\ \ \mathbf{v}\ \ \mathbf{u}] + [\mathbf{v}\ \ \mathbf{w}\ \ \mathbf{w}] + [\mathbf{v}\ \ \mathbf{w}\ \ \mathbf{u}] \\ &= [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{v}\ \ \mathbf{w}\ \ \mathbf{u}]\,. \end{aligned}$$

But the alternating property implies antisymmetry, so

$$[\mathbf{v}\ \ \mathbf{w}\ \ \mathbf{u}] = -[\mathbf{v}\ \ \mathbf{u}\ \ \mathbf{w}] = [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]\,.$$

Therefore, $[(\mathbf{u}+\mathbf{v})\ \ (\mathbf{v}+\mathbf{w})\ \ (\mathbf{w}+\mathbf{u})] = [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{v}\ \ \mathbf{w}\ \ \mathbf{u}] = [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] + [\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}] = 2\,[\mathbf{u}\ \ \mathbf{v}\ \ \mathbf{w}]$.