LINEAR ALGEBRA (MATH 3130): REVIEW SHEET

From the book: Sections 3.3, 4.1–4.7, 5.1–5.5, 5.7, 6.1–6.3, 6.5.

VIII. The determinant. (Continued.)

- (h) Cramer's Rule for solving a linear system $A\mathbf{x} = \mathbf{b}$ with invertible A.

IX. Abstract vector spaces.

- (a) Meaning of the word "abstract".
- (b) Definition and examples of abstract vector spaces, e.g., $M_{m\times n}(\mathbb{R})$, $\mathbb{P}_n(t)$, $C^k([0,1])$. We computed that $M_{m\times n}(\mathbb{R})$, has dimension mn, $\mathbb{P}_n(t)$ has dimension $n+1$, and that $C^0([0,1])$ must be infinite dimensional.
- (c) Coordinates relative to a basis.
- (d) Definition of "isomorphism" of vector spaces. Proof that every finitely generated real vector space is isomorphic to $\mathbb{R}^n$ for some finite n.
- (e) Matrices, ${}_{\mathcal{C}}[T]_{\mathcal{B}}$, for linear transformations between abstract vector spaces. Change of basis matrices, ${}_{\mathcal{C}}[I]_{\mathcal{B}}$.

X. Eigenvalues, eigenvectors, eigenspaces.

- (a) Eigenvectors identify "preserved directions" of a linear transformation $T\colon V \to V$.
- (b) Definitions of eigenvector, eigenvalue, eigenspaces.
- (c) Methods of calculation: characteristic polynomial $\chi_A(\lambda)$ equals $\det(A-\lambda I)$; e-values of A are the roots of $\chi_A(\lambda) = 0$; e-space V_λ equals $\mathrm{Nul}(A-\lambda I)$; λ-eigenvectors are the nonzero vectors of V_λ. Fast calculation of e-values for (block) triangular matrices.
- (d) Complex e-values and e-vectors.

XI. Diagonalization.

- (a) Structure of roots of a real polynomial over $\mathbb{R}$ or $\mathbb{C}$, and of a complex polynomial over $\mathbb{C}$. Algebraic multiplicity of an e-value.
- (b) Geometric multiplicity of an e-value.
- (c) Defn. of "diagonalizable". Thm. A transformation $T\colon V \to V$ is diagonalizable iff V has a basis consisting of e-vectors for T iff the geometric multiplicity of each e-value equals its algebraic multiplicity.
- (d) Independence of subspaces. Sums of subspaces and direct sums of independent subspaces. A sum of distinct e-spaces is direct. $T\colon V \to V$ is diagonalizable iff $V = \bigoplus_\lambda V_\lambda$. (Side observation: $\dim(U \oplus W) = \dim(U) + \dim(W)$.)
- (e) Similarity: A is similar to B if A is a conjugate of B, i.e., $A = S^{-1}BS$. Similarity is an equivalence relation on the set of $n \times n$ matrices. Matrices are similar iff they represent the same transformation relative to different bases. Similar matrices have the same characteristic polynomial, hence same e-values. If $A = S^{-1}BS$, then $S\colon V_\lambda^A \to V_\lambda^B$ is an isomorphism for each e-value λ. A is diagonalizable iff it is similar to a diagonal matrix.
- (f) Cayley-Hamilton Theorem. Minimal polynomial of a matrix. Relationship between the factorizations of $\chi_A(t)$ and $\min_A(t)$. A matrix is diagonalizable iff its minimal polynomial factors into distinct linear factors.
- (g) Applications of diagonalization to the solution of a homogeneous linear ODE with constant coefficients.

XII. Orthogonality.

- (a) Dot product. (Defn. Arithmetic facts follow from those of matrices, since $\mathbf{u} \bullet \mathbf{v} = \mathbf{u}^T\mathbf{v}$.)
- (b) Length in $\mathbb{R}^n$. Unit vector in direction $\mathbf{v}$ is $\mathbf{v}/\|\mathbf{v}\|$.
- (c) Angle in $\mathbb{R}^n$ via $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos(\theta)$.
- (d) Orthogonality. Orthogonal complement. $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$. Algorithm for finding the orthogonal complement of a set of vectors. $U^{\perp\perp} = U$ if $U \leq \mathbb{R}^n$.

(e) Approximate solutions to $A\mathbf{x} = \mathbf{b}$ via least squares. Normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$. Fitting curves to data.
(f) Orthogonal projection onto a subspace.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) From the book: Relevant supplementary problems for chapters 4–6 (excluding problems marked [**M**]).
(2) Computational problems:
  (a) Find bases for the null space, row space and column space of the 3×3 matrix whose entries are all 1. What are the dimensions of these spaces?
  (b) Find a change of basis matrix from $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ and $\mathcal{C} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.
  (c) Find the characteristic equation, e-values, and e-spaces of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Find a matrix S that conjugates A into diagonal form.
  (d) Find a basis for $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}^{\perp}$.
(3) Can any of the following exist? (If so, give an example, if not give a reason.)
  (a) A matrix whose row space is isomorphic to its column space.
  (b) A matrix whose characteristic polynomial is $\lambda^2 + \lambda + 1$.
  (c) An eigenvalue whose geometric multiplicity exceeds its algebraic multiplicity.
  (d) A real 10×10 matrix with only one eigenvector.
  (e) Conjugate matrices of different ranks.
  (f) Conjugate matrices that are not similar.
  (g) A real matrix that is diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.
  (h) A nondiagonalizable complex matrix.
  (i) Subspaces U and W such that $U + W \neq U \oplus W$.
  (j) A matrix whose minimal polynomial is a constant polynomial.
  (k) A matrix whose minimal polynomial has a repeated root.
  (l) A real vector that is orthogonal to itself.
(4) Give the dimensions of the following real vector spaces.

(a) The space of real polynomials $p(t)$ of degree at most 3 which satisfy $p(1) = p(-1) = 0$.
(b) The space of 3×3 upper triangular real matrices.
(c) The space of twice continuously differentiable functions $y = f(x)$ satisfying $y'' = 0$.

(5) How would you solve the following problem? Suppose that V has basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and U is a subspace of V with basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. How do you find a basis for V whose first m vectors form a basis for U?

(6) Suppose that you are given bases $\mathcal{B}$ and $\mathcal{C}$ for subspaces U and W of a space V. How would you find a basis for $U + W$? How would you find a basis for $U \cap W$? (Hint: in both cases, you should apply Gaussian Elimination to the matrix $[\mathcal{B}|\mathcal{C}]$. How should you use the results?)

(7) (Corrected!) Let S be a 2×2 invertible real matrix. Consider the linear transformation of "conjugation by S":

$$T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})\colon A \mapsto S^{-1}AS.$$

Show that if $S\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{u} \neq 0$, and $S^T\mathbf{v} = \mu\mathbf{v}$, $\mathbf{v} \neq 0$, then $A = \mathbf{u}\mathbf{v}^T \neq 0$ has the property that $T(A) = (\mu/\lambda)A$. Do you have any conjectures about the set of e-values of T?

(8) What is the characteristic polynomial for the $n \times n$ matrix whose entries are all 1?

(9) Show that $\dim(U+W) = \dim(U)+\dim(W)-\dim(U\cap W)$. (Solution 1 hint: choose a basis for $U\cap W$ and extend it in different ways to bases for both U and W. Show that all the vectors together form a basis for $U + W$.) (Solution 2 hint: let $\mathcal{B}$ and $\mathcal{C}$ be bases for U and W. Apply the rank+nullity theorem to the matrix $[\mathcal{B}|\mathcal{C}]$.)

(10) The points $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ are the vertices of a regular tetrahedron. Find the lengths of the sides and the angles formed by adjacent faces.

(11) Find the least squares curve of the form $y = ax^2 + bx + c$ that best fits the data points $(-2, 0), (-1, 0), (0, 0), (1, 1), (2, 2)$.

(12) Show that if V is finite dimensional and U is a subspace, then $V = U \oplus U^\perp$.

(13) Show that $(U + W)^\perp = U^\perp \cap W^\perp$.

(14) Explain why $|\mathbf{u} \bullet \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$ whenever $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

(15) Find the minimal polynomial of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$. Is A diagonalizable? What are its e-values?

(16) Find the general solution to $y'' - y' - 2y = 0$.