LINEAR ALGEBRA (MATH 3130): REVIEW SHEET

From the book: Sections 1.1–1.9, 2.1–2.4, 2.7–2.9, 3.1–3.3.

I. Systems of linear equations.
   (a) Augmented matrix and coefficient matrix of a system.
   (b) Row reduction. (Reduced) row echelon form. Pivot positions, pivot columns.
   (c) Free and basic(=pivot) variables. Solutions sets. Parametrized form of a solution.
   (d) Consistent and inconsistent systems.
   (e) Homogeneous systems. Relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of $A\mathbf{x} = \mathbf{0}$.

II. Matrix arithmetic.
   (a) Matrices can be added, negated, multiplied with each other, and scaled, provided the dimensions are right.
   (b) Multiplication of $n \times n$ matrices is not commutative if $n > 1$.
   (c) Matrix transpose.
   (d) Left and right inverses. (Equivalent properties.) Two-sided inverses. A 1-sided invertible matrix is 2-sided invertible iff it is square.
   The following are equivalent to the left invertibility of A:
      (i) $T(\mathbf{x}) = A\mathbf{x}$ is a 1-1 linear transformation.
      (ii) The set of columns of A is independent.
      (iii) Every column of A contains a pivot.
      (iv) $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
   The following are equivalent to the right invertibility of A:
      (i) $T(\mathbf{x}) = A\mathbf{x}$ is an onto linear transformation.
      (ii) The set of columns of A spans $\mathbb{R}^m$.
      (iii) Every row of A contains a pivot.
      (iv) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
   (e) Algorithm for finding inverses.
   (f) Partitioned matrices. Block diagonal and block triangular matrices.

III. Vectors and vector spaces.
   (a) Linear systems may be viewed as vector equations.
   (b) ~~Definition of vector space.~~ Definition of subspace.
   (c) Geometric interpretation of vector space operations.
   (d) Definition of column space and nullspace of a matrix.
   (e) Spanning set of vectors.
   (f) Linearly (in)dependent set of vectors.

IV. Linear Transformations.
   (a) Definition. Fact that any linear transformation has the form $T(\mathbf{x}) = A\mathbf{x}$.

(b) The problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ may be viewed as the problem of finding a vector $\mathbf{x} \in T^{-1}(\mathbf{b})$ for $T(\mathbf{x}) = A\mathbf{x}$.
(c) One-to-one and onto transformations.
(d) Finding the standard matrix of a transformation.
(e) Matrices for rotation and reflection in the plane.

V. Applications.
(a) Balancing a chemical reaction.
(b) Network flow.
(c) Nutritional diet.

VI. Affine transformations.
(a) An affine transformation has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{t}$. Examples: translations, plane reflections about an arbitrary line, and plane rotations about an arbitrary point.
(b) Affine transformations in $\mathbb{R}^n$ may be represented in homogeneous coordinates in $\mathbb{R}^{n+1}$ by matrices of the form $\left[\begin{array}{c|c} A & \mathbf{t} \\ \hline \mathbf{0} & 1 \end{array}\right]$.

VII. Subspaces
(a) The structure of subspaces of $\mathbb{R}^n$.
(b) Four fundamental subspaces: $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Row}(A) = \mathrm{Col}(A^T)^T$, and $\mathrm{Nul}(A^T)^T$.
(c) Ordered and unordered bases for a subspace.
(d) We gave algorithms for finding bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.
(e) Standard basis for $\mathbb{R}^n$.
(f) Dimension of a subspace.
(g) Rank and nullity of a matrix. Rank + nullity theorem.
(h) Proof that "dimension" is well defined, namely, that the size of any independent set is less or equal the size of any spanning set, ~~and that a maximal independent set is spanning while a minimal spanning set is independent.~~

VIII. The determinant.
(a) Signed volume.
(b) Minor, cofactor, definition of the determinant via the Laplace expansion.
(c) $\det(A)$ is defined only if A is square. $\det(A) \neq 0$ iff the columns of A are independent.
(d) Adjugate matrix. Fact that $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$, hence $A^{-1} = (1/\det(A))\mathrm{adj}(A)$ when A is invertible.
(e) Further properties: $\det(AB) = \det(A)\det(B)$, the determinant can be computed by Gaussian elimination, the determinant of a block triangular matrix is the product of the determinants of the blocks, if $T(\mathbf{x}) = A\mathbf{x}$, then the determinant of A measures the "volume expansion" associated with T.
(f) "Correct" definition: the determinant is the unique alternating multilinear function d of n variables defined on $\mathbb{R}^n$ for which $d(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

(g) Permutation expansion of the determinant. Fact that $\det(A) = \det(A^T)$.
(h) ~~Cramer's Rule for solving a linear system $A\mathbf{x} = \mathbf{b}$ with invertible A.~~

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Chapter 1 supplementary problems (excluding problems marked [**M**]).

(2) Chapter 2 supplementary problems (excluding problems marked [**M**]).

(3) Chapter 3 supplementary problems (excluding problems marked [**M**]).

(4) Find the equation of a circle in the plane that passes through the points $(-3, 1), (-1, 5), (5, 5)$. (Hint: set up a linear system.)

(5) Let A be a square matrix. Explain why if the columns of A are independent, then the columns of A^2 are independent.

(6) Show that if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, then $\{\mathbf{a}, \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}+\mathbf{c}\}$ is also independent.

(7) Explain why if A and B are $n \times n$ matrices satisfying $A\mathbf{x} = B\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

(8) Use the definition of "linear transformation" to show that the composition of two linear transformations is a linear transformation.

(9) Explain why if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subset of $\mathbb{R}^n$ that is linearly independent and spans the space, then $k = n$.

(10) Matrices A and B commute if $AB = BA$. Which matrices commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$? (Hint: set up a linear system.)

(11) Explain why the set of columns of an $n \times n$ invertible matrix spans $\mathbb{R}^n$. Then explain why this set of columns is independent.

(12) An $n \times n$ matrix M is symmetric if $M^T = M$, and is antisymmetric if $M^T = -M$.

(a) Show that $S = \frac{1}{2}\left(M + M^T\right)$ is symmetric, $A = \frac{1}{2}\left(M - M^T\right)$ is antisymmetric, and $M = S + A$.

(b) Show that there is exactly one way to write a square matrix M as $S + A$ with S symmetric and A antisymmetric.

(13) Computational problems:

(a) Using homogeneous coordinates, find a matrix representation for the transformation that rotates the plane 45° around the point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Find bases for the null space, row space and column space of the 3×3 matrix whose entries are all 1. What are the dimensions of these spaces?

(c) Put the numbers $1, 2, \ldots, 9$ into a 3×3 matrix in order. What is the determinant?

(14) Can any of the following exist? (If so, give an example, if not give a reason.)

(a) A vector space with an empty basis.

(b) A matrix of rank zero.

(c) A matrix with no determinant.

(d) An invertible matrix whose row sums are all zero.

(e) A real matrix whose null space equals its column space.

(f) A matrix where the dimension of the row space is greater than the dimension of the column space.

(g) A real number that does not arise as the determinant of a real matrix.

(h) A vector space with no subspaces.

(i) A matrix equal to its adjugate.