Rings, ideals

Modern Algebra 1

Fall 2016

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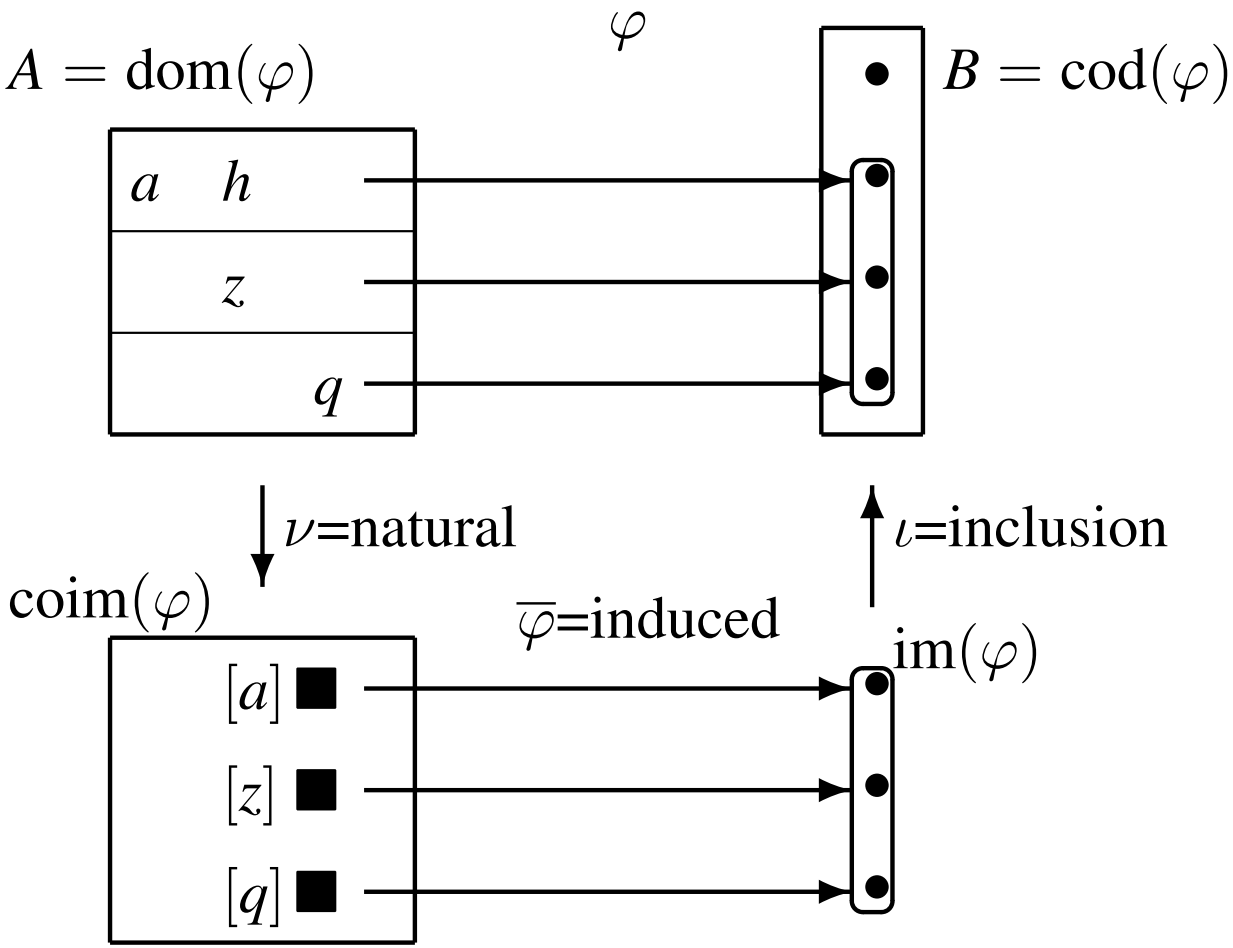

We know what "homomorphism, kernel, image, coimage, subring, and quotient ring" should mean.

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3. For any field $\mathbb{F}$, there is at most one homomorphism $\varphi\colon \mathbb{Q} \to \mathbb{F}$.

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So, $I \lhd R$ iff $\exists S, \exists \varphi(\varphi^{\mathsf{hom}}\colon R \to S)$ such that $I = \varphi^{-1}(0)$.

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Furthermore, ideals under product forms a commutative monoid with unit $(1) = R$.

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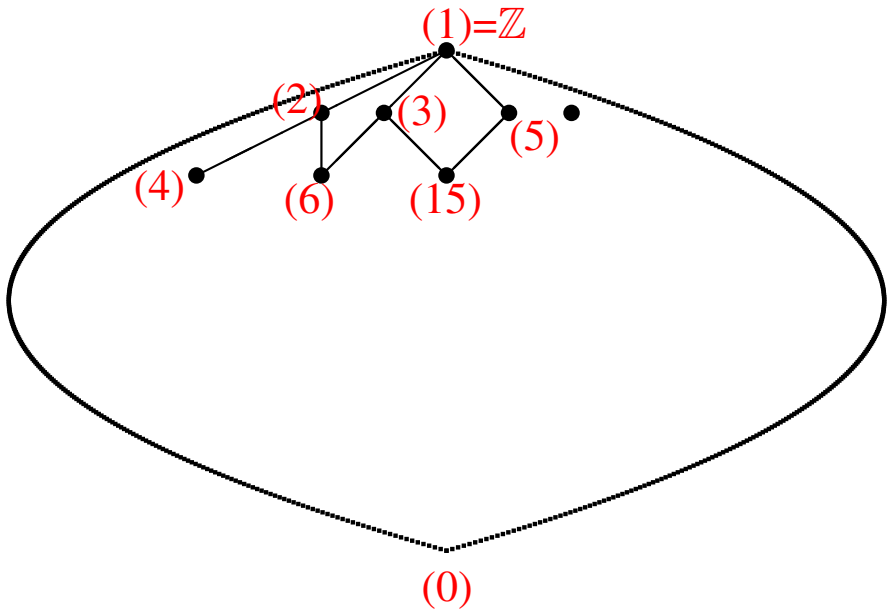

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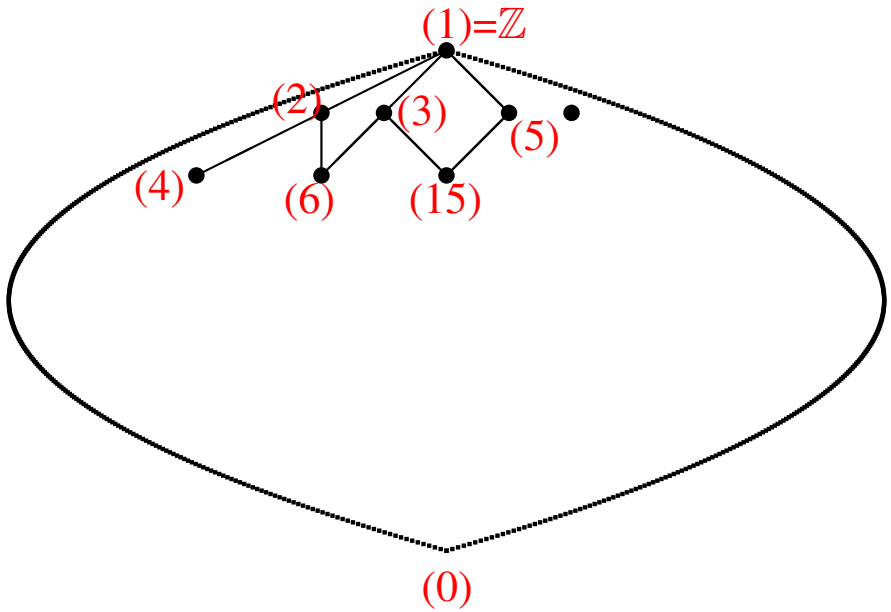

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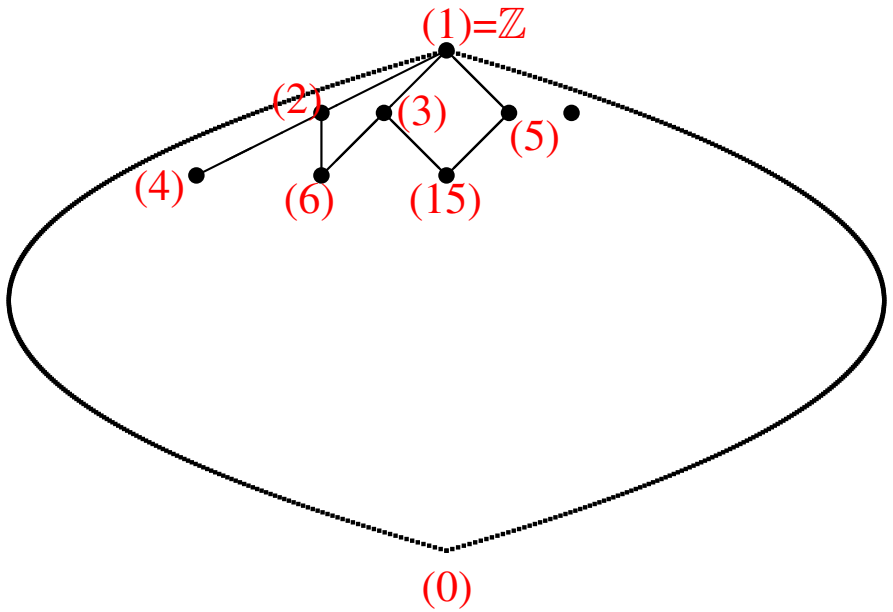

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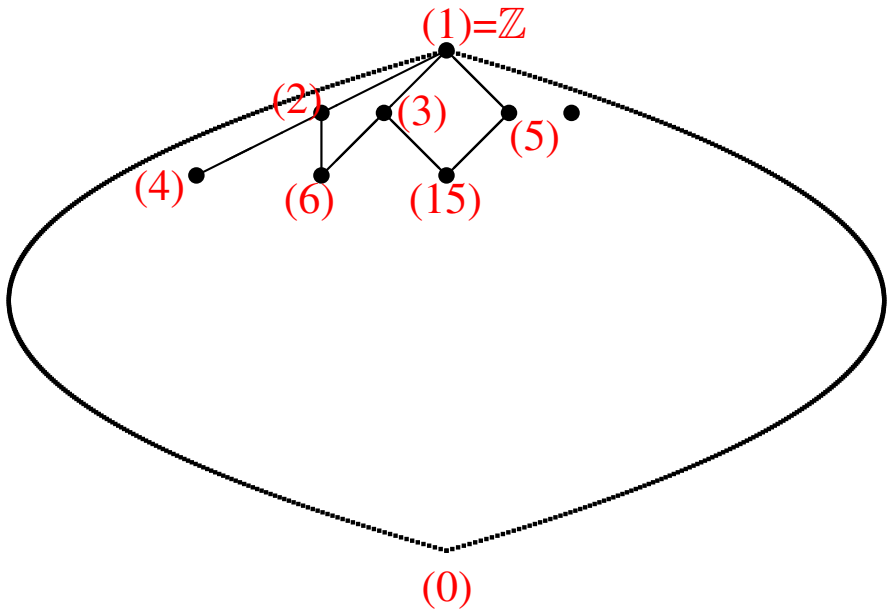

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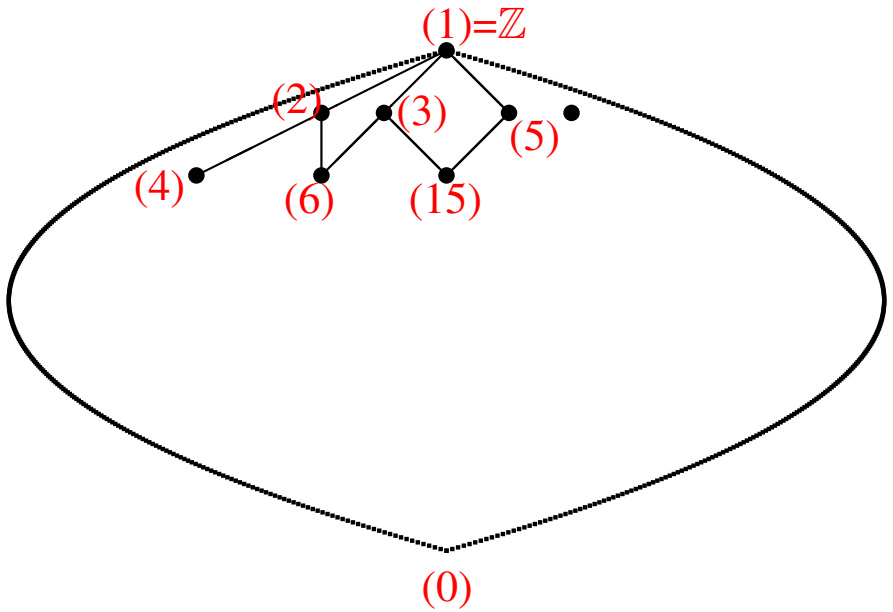

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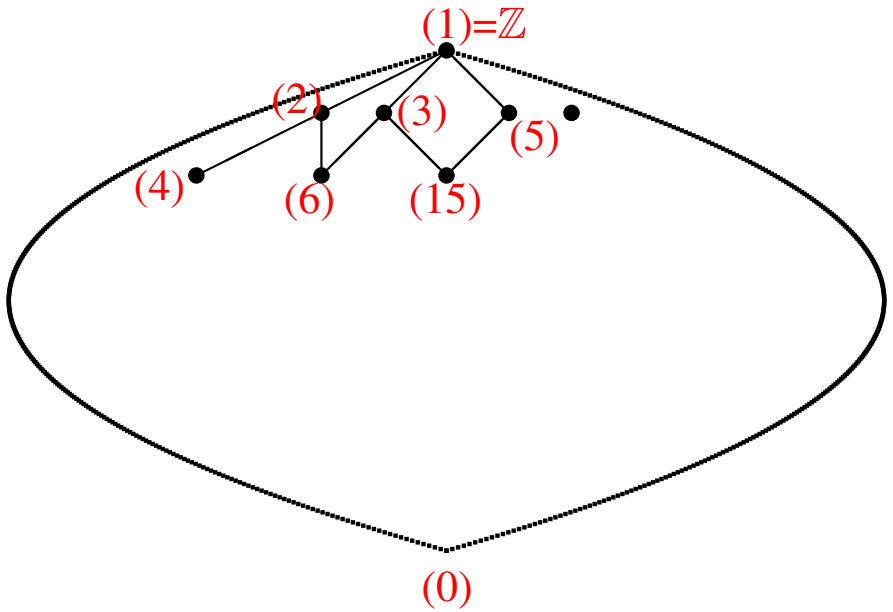

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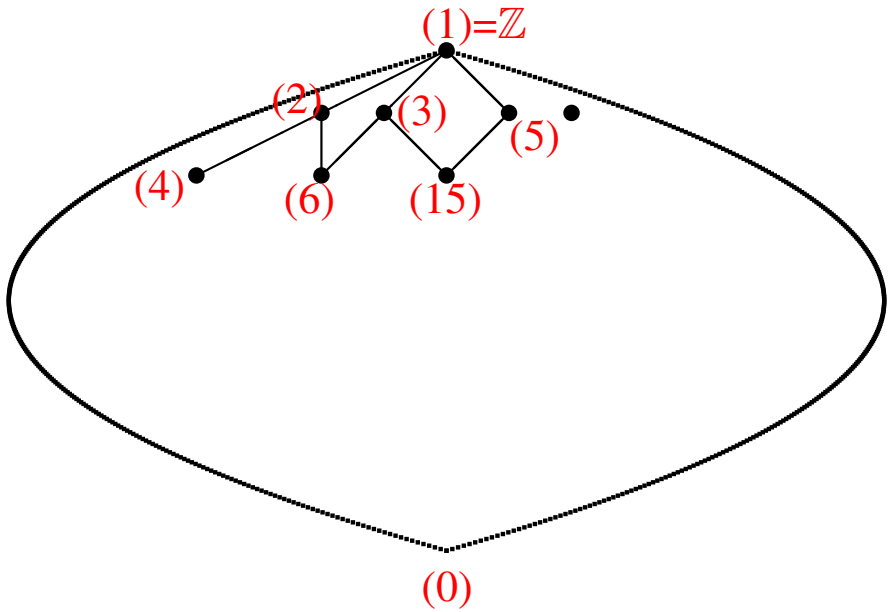

1. $\forall I \lhd \mathbb{Z}$, $I = (n)$ for some $n \in \mathbb{Z}_{\geq 0}$
2. Inclusion order is reverse of divisibility: $(n) \subseteq (m)$ iff m divides n.
3. Maximal ideals are (p) for p prime.
4. $(m) + (n) = (d)$ for $d = \gcd(m, n)$.
5. $(m) \cap (n) = (\ell)$ for $\ell = \mathrm{lcm}(m, n)$.
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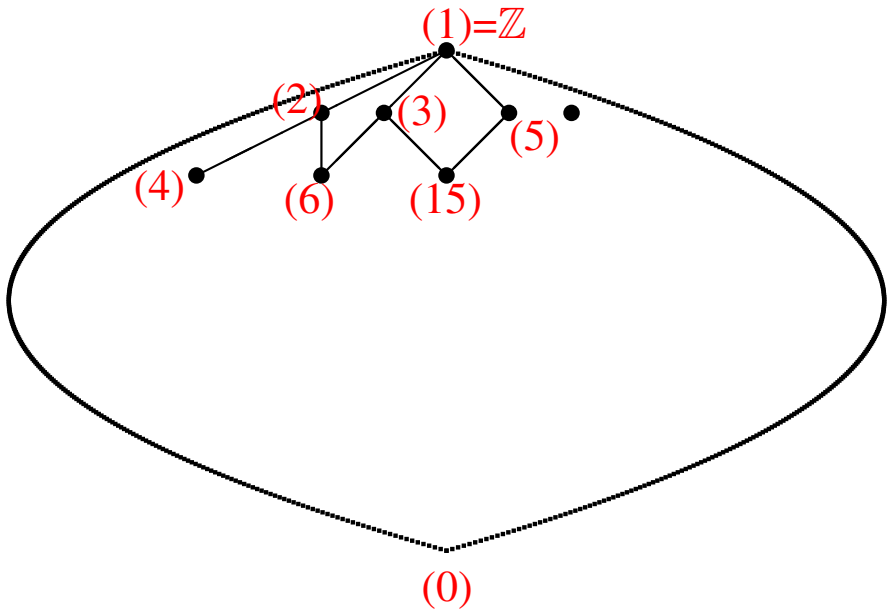

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7. (m) and (n) *comaximal* $\overset{df}{\iff} (m) + (n) = (1) \iff \gcd(m, n) = 1$.

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Can extend to more congruences using above Facts.

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