Key facts about the determinant.

- **Computing the determinant**
  (1) The permutation expansion: $\det([a_{ij}]) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n}$.
  (2) The cofactor expansion:
  $\det(A) = (-1)^{i+1} a_{i1} A_{i1} + \cdots + (-1)^{i+j} a_{ij} A_{ij} + \cdots + (-1)^{i+n} a_{in} A_{in}$
  where A_{ij} is the determinant of the matrix obtained from $A = [a_{ij}]$ by deleting the i-th row and the j-th column. You can expand along any row or column.
  (3) Row reduction.
- **Properties of the determinant**
  (1) The determinant is the only multilinear alternating function of the columns of $n \times n$ matrices whose value on the identity matix is 1.
  (2) The determinant is an antisymmetric function of the columns.
  (3) $\det(AB) = \det(A)\det(B)$.
  (4) The determinant of a triangular matrix is the product of the diagonal entries. (The determinant of a block triangular matrix is the product of the determinants of the diagonal blocks.)
  (5) $\det(A^{-1}) = \frac{1}{\det(A)}$.
  (6) $\det(A^T) = \det(A)$.
- **Applications of the determinant**
  (1) Testing for invertibility.
  (2) Computing A^{-1}. Given $A = [a_{ij}]$, the *adjugate matrix* of A is defined to be: $\operatorname{adj}(A) = [(-1)^{i+j} A_{ij}]^T$ where A_{ij} is the determinant of the matrix obtained from A by deleting the i-th row and the j-th column. The main property of the adjugate that makes it interesting is:
  $$(\operatorname{adj}(A)) \cdot A = A \cdot (\operatorname{adj}(A)) = \det(A) \cdot I.$$
  (3) Cramer's rule for solving systems.
  (4) Computing volume.