Free Algebras, especially groups

Modern Algebra 1

Fall 2016

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The universal property is established like this: given a set-function $X \to A : x_i \mapsto a_i$, extend it to an algebra homomorphism $\mathbb{F}_{\mathcal{C}}(X) \to \mathbb{A} : w(x_1, \ldots, x_n) \mapsto w(a_1, \ldots, a_n)$ (evaluate the word $w(\bar{x})$ at $\bar{a}$).

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Now let $\alpha = \alpha_1 \cdots \alpha_k$, $k \geq 1$, $\beta = \beta_1 \cdots \beta_\ell$, $\ell \geq 1$, $|\gamma| = 1$, and use induction on $|\beta|$. Write $\alpha \top \beta$ to mean $\alpha_k = \beta_1^{-1}$, $\alpha \perp \beta$ to mean $\alpha_k \neq \beta_1^{-1}$, α_* to mean α minus last letter, ${}_*\beta$ to mean β minus first letter, etc.

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That is, "free basis" = "independent generating set".

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To see that G is independent, we must show that any relation $w(\bar{g}) = w'(\bar{g})$ that holds for a tuple of distinct elements in G holds universally. It suffices to show that if $g \neq h^{-1} \in G \cup G^{-1}$, then one can determine g and h from an examination of gh. (More specifically, show that g and gh have the same left two blocks when $g \neq h^{-1}$.)