Universal Properties, Free Algebras, Presentations

Modern Algebra 1

Fall 2016

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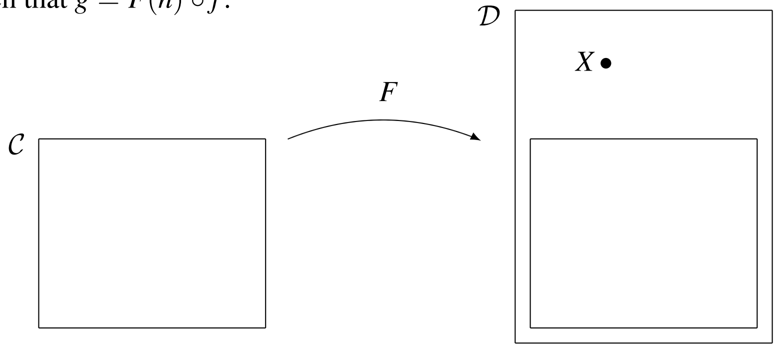

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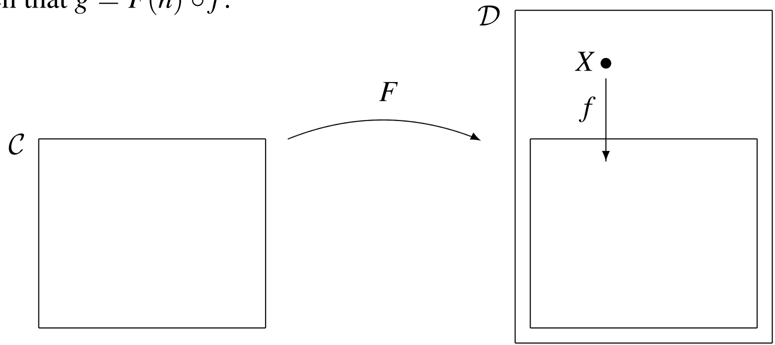

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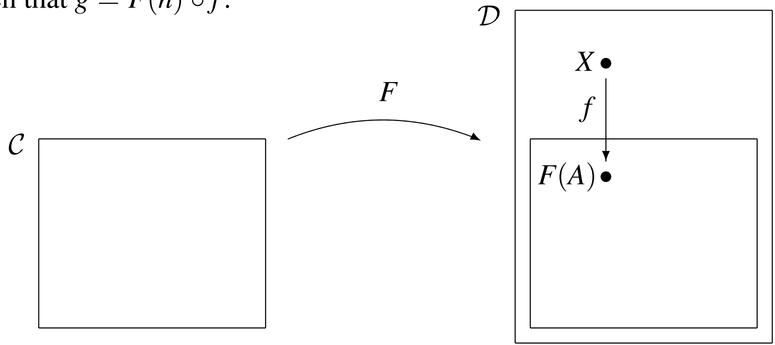

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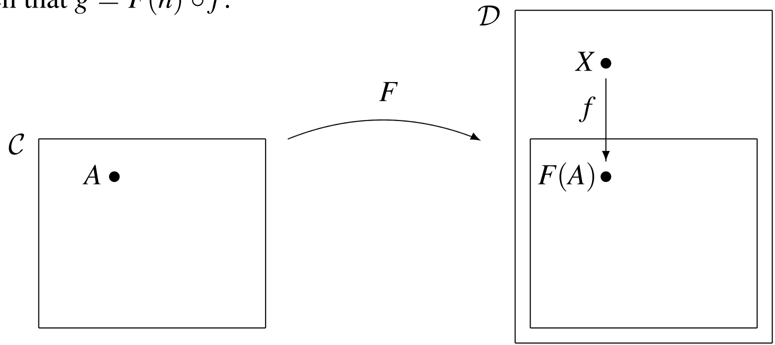

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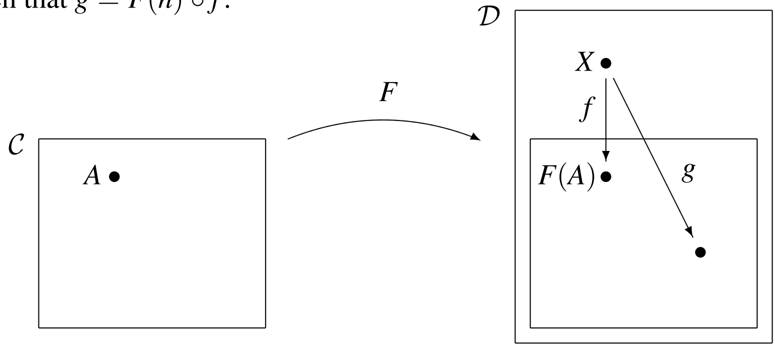

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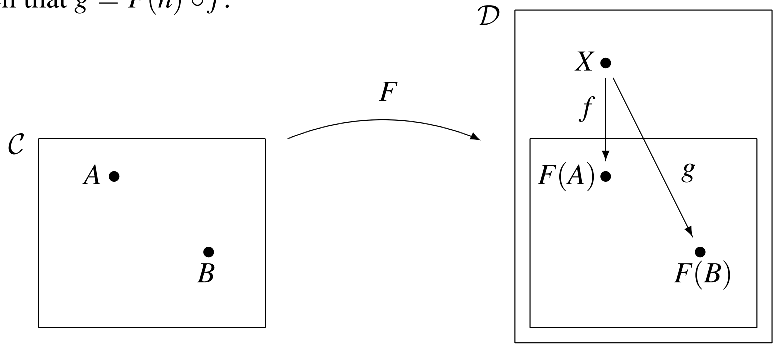

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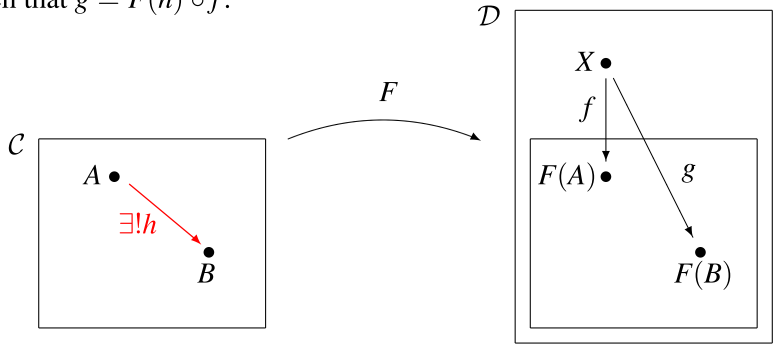

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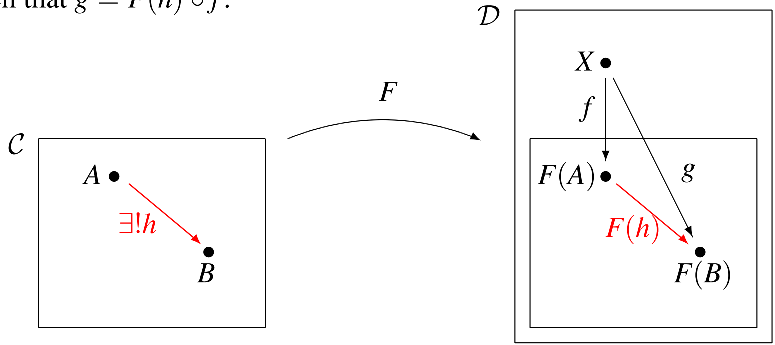

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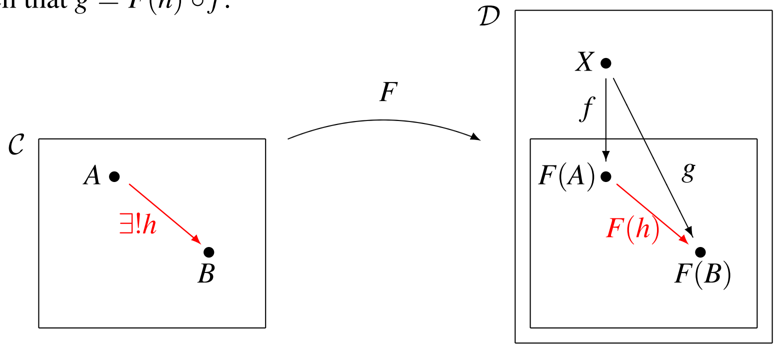

The universal property of (A, f) is the statement that it is a universal arrow.

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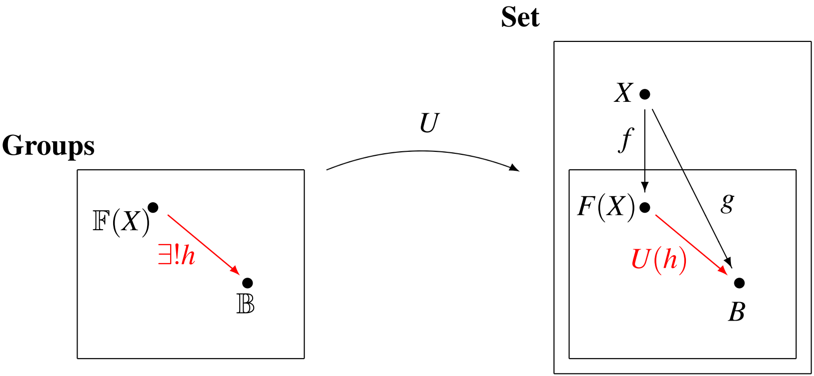
(3) If $U\colon \mathcal{C} \to \mathit{Set}$ is the forgetful functor, then the statement that a universal arrow (F, ι) exists from X to U is the statement that F is free over X and $\iota\colon X \to F$ is *insertion of generators*.

Reinforcement!



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(5) A free commutative ring over X is just the integral polynomial ring $\mathbb{Z}[X]$.

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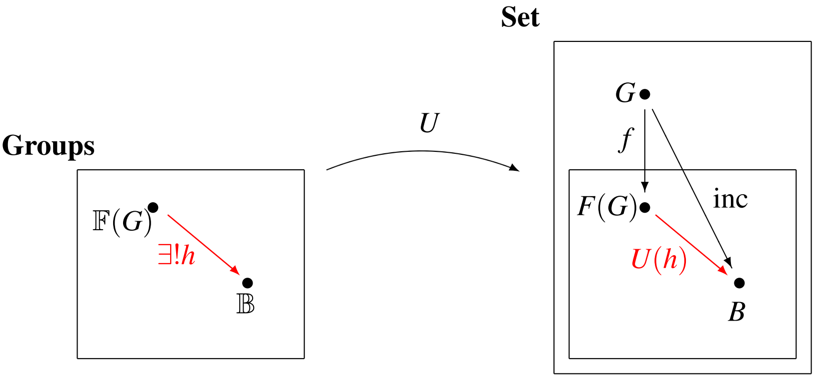
(3) The free topological spaces are the discrete spaces.

(4) The free ordered sets are the discrete orders.

(5) A free commutative ring over X is just the integral polynomial ring $\mathbb{Z}[X]$.

(6) But we want to talk about free groups and presentations.

Presentations of groups



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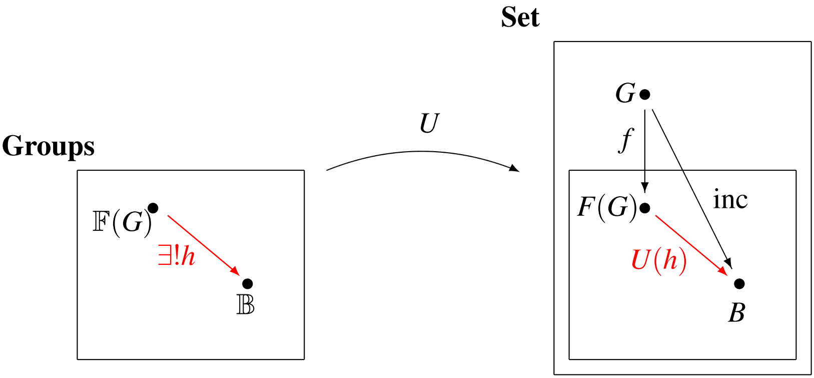

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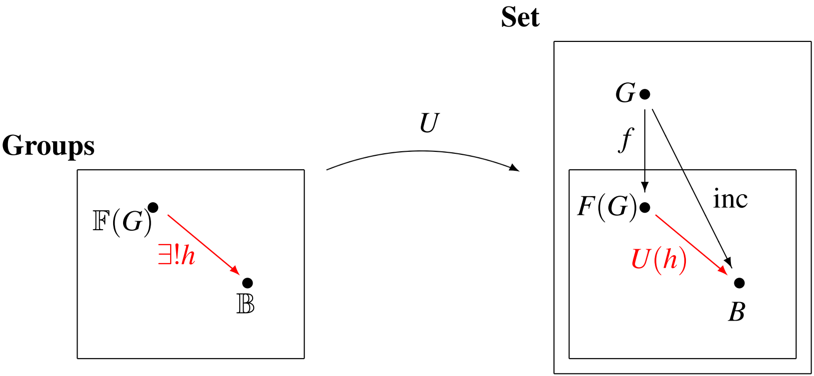

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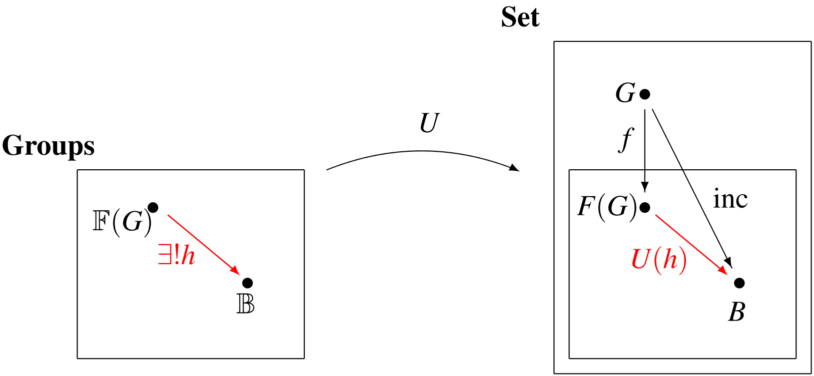

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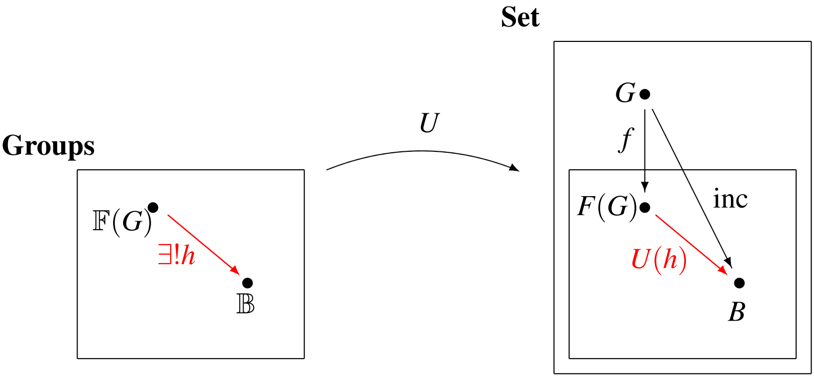

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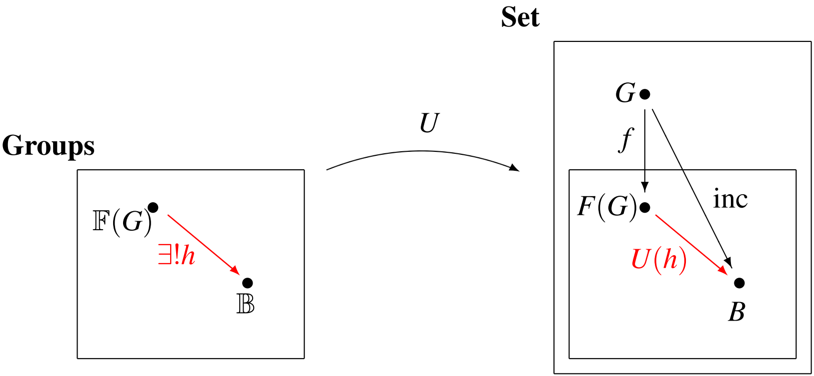

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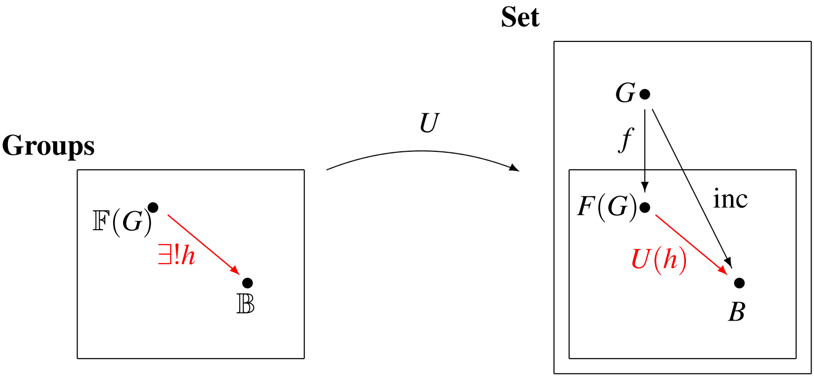

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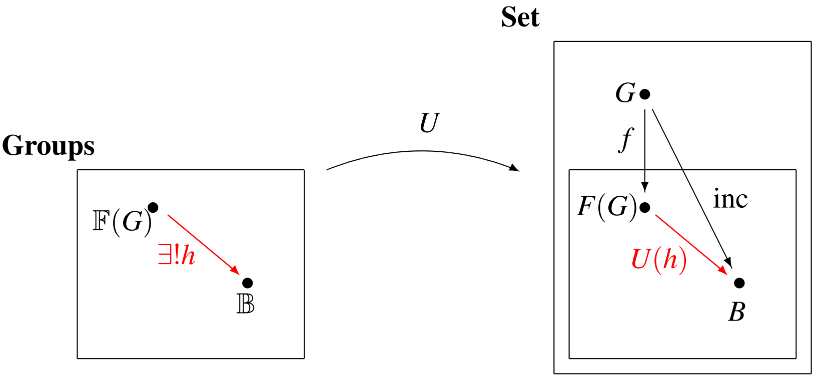

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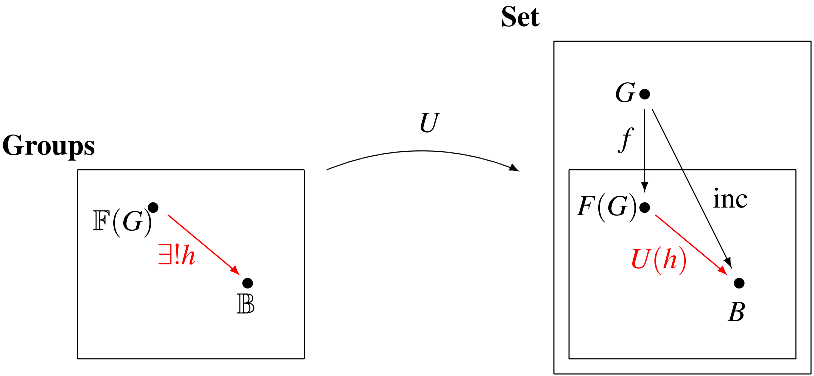

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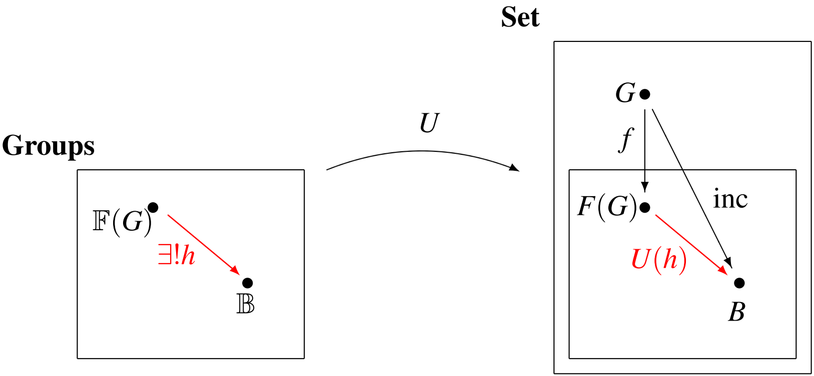

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$= \mathbb{F}(r, f)/\mathrm{cg}(R) = D_n$, where $R = \{(r^n, 1), (f^2, 1), (rf = fr^{-1})\}$.

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$(X \cup X^{-1})^* / \equiv$ is an group satisfying all identities that hold in $\mathcal{C}$. In fact, if $(X \cup X^{-1})^* / \equiv$ belongs to $\mathcal{C}$, then it is the free algebra over X in $\mathcal{C}$. $(X \cup X^{-1})^* / \equiv$ will belong to $\mathcal{C}$ if $\mathcal{C}$ is definable by identities.

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Thus the free group over X may be constructed as the set of reduced words over X, with product defined by: "concatenate words, then reduce". That is, $fg = h$ if, as reduced words, $f = ws$, $g = s^{-1}w'$, $h = ww'$, and the last symbol of w is not the inverse of the first symbol of w'.

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(Finite if…)

The universal property for presentations is derivable by combining the universal property for free algebras with the universal property for quotients. It says:

$\mathbb{A} = \langle G \mid R \rangle$ is the algebra (equipped with a map $f\colon G \to A$) such that for any set-map $g\colon G \to B$ *whose image satisfies the relations* there is a unique algebra homomorphism $h\colon \mathbb{A} \to \mathbb{B}$ such that $h \circ f = g$.

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Example: $\mathbb{Z}_2 \sqcup \mathbb{Z}_2 = \langle x, y \mid x^2 = 1, y^2 = 1 \rangle \cong D_\infty$.