ALGEBRA 1

HOMEWORK ASSIGNMENT VII

(Turn in underlined problems.)

Read Sections 6.3.

SECTION	**PROBLEMS**
6.3	**1, 4, 13**

ADDITIONAL PROBLEMS

1. Show that, for any field $\mathbb{F}$, that the group of invertible upper triangular $n \times n$ matrices over $\mathbb{F}$ is solvable, and the subgroup consisting of the unipotent matrices is nilpotent.

2. (Only parts (c) and (d) are to be handed in.) A *Coxeter group* is a group that has a presentation of the form $\langle x_1, \ldots, x_n \mid (x_i x_j)^{m_{ij}} = 1\rangle$ where (i) the m_{ij} are from $\{1, 2, 3, \ldots, \infty\}$, (ii) $m_{i,i} = 1$ for all i, and (iii) $m_{ij} > 1$ for all $i \neq j$. The diagonal relations $(x_i x_i)^1 = 1$ express that the generators have exponent 2. The exponents are often encoded in the *Coxeter matrix*, $[m_{ij}]$, which may be taken to symmetric (since $(x_i x_j)^m = 1$ iff $(x_j x_i)^m = 1$).

(a) Show that if $m_{ij} = 2$, then the generators x_i and x_j will commute in the presented group.

(b) Show that the Coxeter group with matrix $\begin{bmatrix} 1 & n \\ n & 1 \end{bmatrix}$ is the dihedral group of symmetries of a regular n-gon.

(c) Argue that the Coxeter group with matrix $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ is isomorphic to S_4. (This is the group with generators x_1, x_2, x_3 subject to only the relations $x_i^2 = 1$ for all i, $(x_1 x_2)^3 = 1$, $(x_2 x_3)^3 = 1$, and $x_1 x_3 = x_3 x_1$.)

(d) **Argue that the Coxeter group with matrix** $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ **is infinite by finding a** 3**-generated infinite group satisfying all of the relations. To find such a group, consider the the group of rigid motions of the plane generated by reflections through lines that determine the sides of an equilateral triangle.**

3. Show that any finite group has a finite presentation.

4. Let F **be the free group over** $\{x, y\}$**. Show that** $\{y^{-n}xy^n \mid n \geq 1\}$ **is an independent set, and hence generates a free subgroup of** F **of infinite rank.**