ALGEBRA 1
HOMEWORK ASSIGNMENT IV

(Turn in underlined problems.)

Read Sections 3.4, 3.5, Appendix II.1.

SECTION	PROBLEMS
3.2	**9, 14**
3.3	**4**
3.4	**8, 12**

ADDITIONAL PROBLEMS

1. If G is a simple group, what are the possibilities for the lattice of normal subgroups of $G \times G$?

2. Find groups $N \triangleleft G$, $N' \triangleleft G'$, with

(a) $G \cong G'$, $N \cong N'$, but $G/N \ncong G'/N'$.
(b) $G \cong G'$, $G/N \cong G'/N'$, but $N \ncong N'$.
(c) $G/N \cong G'/N'$, $N \cong N'$, but $G \ncong G'$.

3. Find a group G with $G \cong G \times G$.

4. (Parts (a) and (b) will be considered one problem, and parts (c) and (d) will be considered one problem.)

Let A be an abelian group of order mn where $\gcd(m, n) = 1$. By the Chinese Remainder Theorem, there exists an integer a such that $a \equiv 1 \pmod{m}$ and $a \equiv 0 \pmod{n}$. Let $\varphi\colon A \to A\colon x \mapsto x^a$. Show that

(a) φ is an idempotent endomorphism of A. (Idempotence means $\varphi(\varphi(x)) = \varphi(x)$ for all $x \in A$.)
(b) $A \cong \mathrm{im}(\varphi) \times \ker(\varphi)$.
(c) $|\,\mathrm{im}(\varphi)| = m$ and $|\ker(\varphi)| = n$.
(d) A is isomorphic to a product of groups of prime power order.

5. Show that if N is a minimal normal subgroup of a finite group G, then $N \cong S^k$ for some k and some simple group S.

Challenge Problem! (Prize = 1 coin) Is there a group G such that
(a) every proper subgroup of G is finite, and
(b) every finite group is isomorphic to a subgroup of G?