ALGEBRA 1
HOMEWORK ASSIGNMENT II

(Read all problems, and try to solve them. Turn in underlined problems.)

Read Section 2.1–2.5.

<u>SECTION</u>	**<u>PROBLEMS</u>**
1.1	**15, 19, 25, 28**
1.2	**9-13, 15**
1.3	**11, <u>15</u>**
1.4	**4**
1.6	**1, 7, 19, <u>23</u>**
2.1	**1, 2, 6, 11, 15**

<u>ADDITIONAL PROBLEMS</u>

<u>1.</u> Given a subset $T \subseteq S_n$ construct a graph G_T by taking $\{1, \dots, n\}$ to be the vertex set and $\{\{i, j\} \mid (i\ j) \in T\}$ to be the edge set.

(a) Prove that if H is a subgroup of S_n, then the connected components of G_H are complete subgraphs. (Equivalently, any two vertices connected by a path in G_H are connected by a single edge.)

(b) Prove that if $T \subseteq S_n$ is a set of transpositions, then the subgroup of S_n generated by T is all of S_n iff G_T is connected. (Hint for the backward implication: use (a) with H taken to be the subgroup generated by T.)

(c) Conclude that if p is prime, then S_p is generated by any subset of the form $\{\tau, \pi\}$ where τ is a transposition and π is a p-cycle. (Hint: if $\tau = (i\ j)$, then the subgroup generated by $\{\tau, \pi\}$ contains the subset $T \subseteq S_n$ of all permutations of the form $\pi^k \tau \pi^{-k} = (\pi^k(i)\ \pi^k(j))$.)

(d) Show that if n is not prime, then S_n has a subset $\{\tau, \nu\}$ where τ is a transposition, ν is an n-cycle, and $\{\tau, \nu\}$ does not generate S_n.

2. Let $T \subseteq S_n$ be a set of 3-cycles. Define a hypergraph G_T whose vertex set is $\{1, \dots, n\}$ and whose edge set is $\{\{i, j, k\} \mid (i\ j\ k) \in T\}$. Show that the subgroup of S_n generated by T is A_n iff G_T is connected. (A *hypergraph* is

defined like a graph, except edges can have any number of vertices. In this problem, though, all edges have three vertices. A hypergraph is *connected* if any pair of elements a and b are part of a chain $a = u_1, u_2, \ldots, u_n = b$ where any consecutive pair of elements lie in an edge.)

3. Give necessary and sufficient conditions that m and n must satisfy for D_m to be embeddable in S_n.

Challenge Problem! (Prize = 1 forint!) Prove that the number of subgroups of S_n is at least $2^{\lfloor n^2/16 \rfloor}$.