# ALGEBRA 1
# HOMEWORK ASSIGNMENT II

(Read all problems, and try to solve them. Turn in underlined problems.)

Read Section 2.1–2.5.

| SECTION | PROBLEMS |
|---|---|
| **1.1** | **15, 19, 25, 28** |
| **1.2** | **9-13, 15** |
| **1.3** | **11, <u>15</u>** |
| **1.4** | **4** |
| **1.6** | **1, 7, 19, <u>23</u>** |
| **2.1** | **1, 2, 6, 11, 15** |

## ADDITIONAL PROBLEMS

**<u>1.</u> Given a subset $T \subseteq S_n$ construct a graph $G_T$ by taking $\{1, \ldots, n\}$ to be the vertex set and $\{\{i, j\} \mid (i\ j) \in T\}$ to be the edge set.**

**(a) Prove that if $H$ is a subgroup of $S_n$, then the connected components of $G_H$ are complete subgraphs. (Equivalently, any two vertices connected by a path in $G_H$ are connected by a single edge.)**

**(b) Prove that if $T \subseteq S_n$ is a set of transpositions, then the subgroup of $S_n$ generated by $T$ is all of $S_n$ iff $G_T$ is connected. (Hint for the backward implication: use (a) with $H$ taken to be the subgroup generated by $T$.)**

**(c) Conclude that if $p$ is prime, then $S_p$ is generated by any subset of the form $\{\tau, \pi\}$ where $\tau$ is a transposition and $\pi$ is a $p$-cycle. (Hint: if $\tau = (i\ j)$, then the subgroup generated by $\{\tau, \pi\}$ contains the subset $T \subseteq S_n$ of all permutations of the form $\pi^k \tau \pi^{-k} = (\pi^k(i)\ \pi^k(j))$.)**

**(d) Show that if $n$ is not prime, then $S_n$ has a subset $\{\tau, \nu\}$ where $\tau$ is a transposition, $\nu$ is an $n$-cycle, and $\{\tau, \nu\}$ does not generate $S_n$.**

**2. Let $T \subseteq S_n$ be a set of 3-cycles. Define a hypergraph $G_T$ whose vertex set is $\{1, \ldots, n\}$ and whose edge set is $\{\{i, j, k\} \mid (i\ j\ k) \in T\}$. Show that the subgroup of $S_n$ generated by $T$ is $A_n$ iff $G_T$ is connected. (A *hypergraph* is**

**defined like a graph, except edges can have any number of vertices. In this problem, though, all edges have three vertices. A hypergraph is *connected* if any pair of elements $a$ and $b$ are part of a chain $a = u_1, u_2, \ldots, u_n = b$ where any consecutive pair of elements lie in an edge.)**

**3. Give necessary and sufficient conditions that $m$ and $n$ must satisfy for $D_m$ to be embeddable in $S_n$.**

**Challenge Problem! (Prize = 1 forint!) Prove that the number of subgroups of $S_n$ is at least $2^{\lfloor n^2/16 \rfloor}$.**