ALGEBRA 1

HOMEWORK ASSIGNMENT I

(Turn in underlined problems.)

Read Chapter 0 and Sections 1.1-1.6.

PROBLEMS

1. **Let X be a set, and let** $\mathrm{Inject}(X) = \langle\{f \in X^X \mid f \text{ is an injection}\}; \circ, \boldsymbol{id}\rangle$. **Algebraize the theory of injective functions by showing that**

(a) $\mathrm{Inject}(X)$ **satisfies**
  (i) $(f \circ (g \circ h)) = ((f \circ g) \circ h)$,
  (ii) $\boldsymbol{id} \circ f = f \circ \boldsymbol{id} = f$, **and**
  (iii) $(f \circ g = f \circ h) \Rightarrow (g = h)$.
  (I.e., $\mathrm{Inject}(X)$ **is a left cancellative monoid.)**

(b) **If** $\mathrm{M} = \langle M; \cdot, 1\rangle$ **is an algebra satisfying (a)(i)–(iii), then M is embeddable in** $\mathrm{Inject}(X)$ **for some** X**.**

2. **Let** $\mathrm{Surject}(X) = \langle\{f \in X^X \mid f \text{ is a surjection}\}; \circ, \boldsymbol{id}\rangle$. **Algebraize the theory of surjective functions by showing that**

(a) $\mathrm{Surject}(X)$ **is a right cancellative monoid (i.e., satisfies 1(a)(i)-(ii) and**
  (iii)' $(f \circ h = g \circ h) \Rightarrow (f = g)$**.)**

(b) **If** $\mathrm{M} = \langle M; \cdot, 1\rangle$ **is a right cancellative monoid, then M is embeddable in** $\mathrm{Surject}(X)$ **for some** X**.**

Let $\mathrm{Biject}(X) = \langle\{f \in X^X \mid f \text{ is a bijection}\}; \circ, \boldsymbol{id}\rangle$. **It would be natural to guess that** $\mathrm{Biject}(X)$ **is algebraized by the class of (left *and* right) cancellative monoids, but this is wrong. Any algebra of the form** $\mathrm{Biject}(X)$ **is indeed a cancellative monoid, but must satisfy stronger properties that do not follow from cancellativity, such as the one in the next problem.**

3. Show that $\mathrm{Biject}(X)$ **satisfies**

$$(f \circ g = f' \circ g')\&(h \circ g = h' \circ g')\&(h \circ k = h' \circ k') \Rightarrow (f \circ k = f' \circ k').$$

4. Modify Biject(X) **to** $\text{Perm}(X) = \langle\{f \in X^X \mid f \text{ is a bijection}\}; \circ, {}^{-1}, \boldsymbol{id}\rangle$. **Here** f^{-1} **is the inverse of the function** f**. Show that the theory of bijective functions is algebraized by structures** $\mathbf{G} = \langle G; \cdot, {}^{-1}, 1\rangle$ **satisfying**

(i) $(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$**,**
(ii) $1 \cdot x = x \cdot 1 = x$**, and**
(iii) $x \cdot x^{-1} = x^{-1} \cdot x = 1$**.**

That is, show that Perm(X) **satisfies (i)-(iii), and that any structure of type** $\langle 2, 1, 0\rangle$ **satisfying (i)-(iii) is embeddable in** Perm(X) **for some** X**.**

<u>5.</u> Let $\langle P; \leq\rangle$ **be a set equipped with a partial order. Suppose that any two elements of** P **have a greatest lower bound, i.e.**

$$\forall a, b\, \exists c((c \leq a) \,\&\, (c \leq b) \,\&\, \forall d(((d \leq a) \,\&\, (d \leq b)) \rightarrow (d \leq c))),$$

is satisfied. Algebraize this situation by defining a binary operation $*$ **on** P **such that** $a * b$ **equals the greatest lower bound of** a **and** b**. Find identities satisfied by** $*$**, and show your list of identities to be complete by proving a representation theorem. (Your representation theorem should show that any algebra** $\mathbf{A} = \langle A; *\rangle$ **satisfying your identities is embeddable into a partially ordered set in such a way that** $a * b$ **is the greatest lower bound of** a **and** b **with respect to the given order.)**

Challenge Problem! Show that the implication in Problem 3 is not a consequence of the associative laws, the unit laws and cancellativity by exhibiting a cancellative monoid that fails the implication. (Such a cancellative monoid cannot be represented as a monoid of bijections.)