ALGEBRA TEST #2

This exam is due Thursday, December 15, at 10pm. Do three of the problems. You may use your book, but you may not communicate with others concerning the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

I have neither given nor received aid on this exam.

Name:____________________________

1. Show that there is no simple group of order 120.

2. Show that if G is a simple group of order $4pq$, where p and q are distinct odd primes, then $|G| = 60$.

3. Prove that a surjective endomorphism of a Noetherian ring is an isomorphism.

4. Show that the intersection of any chain of prime ideals in a commutative ring is prime. Conclude that if $I \lhd R$ is a proper ideal, then there exists a minimal element in the set of prime ideals containing I.

5. This problem concerns the rational root theorem, specifically the consequence of the RRT that a rational root of a monic integer polynomial is an integer.

If Z is an integral domain contained in a field $\mathbb{F}$, then an element $\alpha \in \mathbb{F}$ is *integral over* D if α satisfies a monic polynomial with coefficients in D. D is *integrally closed* if every element of the field of fractions of D is contained in D.

(a) Keeping the RRT in mind, explain why any UFD is integrally closed.
(b) Show that $Z[\sqrt{-3}]$ is not integrally closed, hence not a UFD. (Hint: first calculate the rooots of $x^2 + x + 1$.)

6. Let $\omega = e^{2\pi i/3}$ be a primitive third root of unity. Show that $\mathbb{Z}[\omega]$ is a Euclidean Domain. Show that $1 - \omega$ is a prime element.

7. Use the fact that $\mathbb{Z}[i]$ is a unique factorization domain to find all integer solutions to $x^2 + y^2 = z^3$ that satisfy $\gcd(x, y) = 1$.

8. Let $\mathbb{F}$ be a field. Show that $\mathbb{F}[x]$ has an infinite set of pairwise nonassociate primes.

9. Decide (with proof) whether r is irreducible in the integral domain $\mathbf{D}$:

(a) $r = x^n + px + p^2$, $\mathbf{D} = \mathbb{Z}[x]$, p is a prime integer.
(b) $r = x^2 + y^2 - 1$, $\mathbf{D} = \mathbb{F}[x, y]$, $\mathbb{F}$ is a field.
(c) $r = x^4 - 10x^2 + 1$, $\mathbf{D} = \mathbb{Z}_p[x]$, p an unspecified prime. (I.e., decide the answer for each p.)