ALGEBRA TEST #1

This exam is due Monday, October 17. Do two of the problems. You may use your book, but you may not communicate with others concerning the exam. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

I have neither given nor received aid on this exam.

Name:____________________________

1. Draw a regular 4-gon and label the vertices with the numbers $1, 2, 3, 4$.

   (a) Write the cycle decomposition of each element of $D_4 = \{1, r, r^2, r^3, f, rf, r^2f, r^3f\}$.

   (b) Is it possible to label the *edges* with $1, 2, 3, 4$ so that the cycle decomposition of each permutation in D_4 is the same as it was in (a)? (Show or explain.)

   (c) For which m (if any) do there exist labelings of the vertices and the edges of a regular m-gon with $1, \ldots, m$ so that the cycle decomposition of any permutation relative to the vertex labeling is the same as the cycle decomposition relative to the edge labeling?

2. If G is a group and $m, n \in G$, then the *commutator of* m *and* n is $[m, n] := m^{-1}n^{-1}mn$. If M and N are subgroups, the *commutator of* M *and* N is subgroup generated by $\{[m, n] \mid m \in M, n \in N\}$.

(a) Show that $[M, N] = \{1\}$ iff every element of M commutes with every element of N.

(b) Show that a subgroup $N \leq G$ is normal iff $[G, N] \subseteq N$.

(c) Show that if N is a normal subgroup of S_n, then either $[S_n, N] = \{1\}$ or N contains an element that is a product of exactly two transpositions. (Use the fact that S_n is generated by transpositions.)

(d) Show that if N is a normal subgroup of S_n, $n > 4$, and N contains an element that is a product of exactly two transpositions, then N contains a 3-cycle.

3. Suppose that $N_1 \lhd G_1$ and $N_2 \lhd G_2$.

(a) Show that the composite homomorphisms $G_1 \times G_2 \xrightarrow{\pi_1} G_1 \xrightarrow{\nu_1} G_1/N_1$ and $G_1 \times G_2 \xrightarrow{\pi_2} G_2 \xrightarrow{\nu_2} G_2/N_2$ induce a homomorphism $G_1 \times G_2 \to (G_1/N_1) \times (G_2/N_2)$ with kernel $N_1 \times N_2$.

(b) Show that the homomorphism in (a) induces an isomorphism from $(G_1 \times G_2)/(N_1 \times N_2)$ to $(G_1/N_1) \times (G_2/N_2)$.

(c) (This part is unrelated to (a) and (b).) Explain how the universal property of products establishes $G_1 \times G_2 \cong G_2 \times G_1$.

4. In a HW problem (HW IV.4) you proved that any finite abelian group is a product of groups of prime power order. These factors of prime power order are called the *primary components* of the group. (A primary component of p-power order is called a p-primary component.)

This problem investigates the structure of the primary components of a finite abelian group.

(a) (Cyclic subgroups of maximal order split off.) Suppose A is finite, abelian, and of prime power order. Let $z \in A$ have maximal order, and let $C = \langle z \rangle$. Let $H \leq A$ be a subgroup maximal for the property that $H \cap C = \{0\}$. Show that H is a complement of C.

(b) Deduce from (a) that A is a product of cyclic groups.

(c) Let $A[p] = \{a \in A \mid pa = 0\}$ be the annihilator of p in A. (A is considered as an additive group.) Show that if $A \cong \mathbb{Z}_{p^{e_1}} \times \cdots \times \mathbb{Z}_{p^{e_k}}$ with $1 \leq e_1 \leq e_2 \leq \cdots \leq e_k$, then $|A[p]| = p^k$ and $A/A[p] \cong \mathbb{Z}_{p^{e_1-1}} \times \cdots \times \mathbb{Z}_{p^{e_k-1}}$.

(d) Deduce from (c) that if A is factored into cyclic subgroups, then the number of factors of order at least p is uniquely determined. Then explain why the number of factors of order at least p^2 is uniquely determined. Then explain why the numbers e_i from (c) are uniquely determined.