Modern Algebra 1 (MATH 6130)
September 16, 2016

Left and right invariant equivalence relations on groups.

Definition 1. An equivalence relation θ on a group G is *left invariant* (LI) if for all $a, b, c \in G$

$$a \equiv b \pmod{\theta} \quad \text{implies} \quad ca \equiv cb \pmod{\theta}.$$

The concept of right invariance is defined the same way, but with c appearing on the right.

We proved the following in class:

Theorem 2. *An equivalence relation θ on a group is the kernel of a homomorphism iff it is left and right invariant.* $\square$

So now let's understand which equivalence relations are left invariant. First, for a subgroup $H \leq G$ define ${}_H\theta = \{(a, b) \in G^2 \mid a^{-1}b \in H\}$.

Theorem 3. *For any $H \leq G$, the relation ${}_H\theta$ is a left invariant equivalence relation on G. Moreover, the ${}_H\theta$-class of 1 is H itself.*

Proof. ${}_H\theta$ is

(1) (reflexive) $(a, a) \in {}_H\theta$ iff $a^{-1}a \in H$. ✓
(2) (symmetric) If $(a, b) \in {}_H\theta$, then $x = a^{-1}b \in H$, so $x^{-1} = b^{-1}a \in H$, so $(b, a) \in {}_H\theta$. ✓
(3) (transitive) If $(a, b), (b, c) \in {}_H\theta$, then $x = a^{-1}b, y = b^{-1}c \in H$, so $xy = a^{-1}c \in H$, so $(a, c) \in {}_H\theta$. ✓
(4) (left invariant) If $(a, b) \in {}_H\theta$, then $a^{-1}b \in H$, so $(ca)^{-1}(cb) = a^{-1}b \in H$, so $(ca, cb) \in {}_H\theta$. ✓

Now observe that the ${}_H\theta$-class of 1 is just the set of all $b \in G$ such that $(1, b) \in {}_H\theta$, which is the set of b such that $1^{-1}b = b \in H$. This shows that $1/{}_H\theta = H$. $\square$

Theorem 4. *If θ is any left invariant equivalence relation on G, then the θ-class of 1 is a subgroup $H \leq G$, and $\theta = {}_H\theta$.*

Proof. Let $H = 1/\theta = \{a \in G \mid a \equiv 1 \pmod{\theta}\}$. This set

(1) (contains 1) since $1 \equiv 1 \pmod{\theta}$. ✓
(2) (closed under inverse) since $a \equiv 1 \pmod{\theta}$ implies $a^{-1}a \equiv a^{-1}1 \pmod{\theta}$ or just $1 \equiv a^{-1} \pmod{\theta}$. ✓
(3) (closed under product) If $a \equiv 1 \pmod{\theta}$ and $b \equiv 1 \pmod{\theta}$, then $ab \equiv a1 = a \equiv 1 \pmod{\theta}$, so $ab \equiv 1 \pmod{\theta}$. ✓

Finally, $(a, b) \in \theta$ iff $a \equiv b \pmod{\theta}$ iff $1 = a^{-1}a \equiv a^{-1}b \pmod{\theta}$ iff $a^{-1}b \in H$ iff $(a, b) \in {}_H\theta$. $\square$

These theorems have consequences, for example:

(1) The maps $\theta \mapsto 1/\theta$, $H \mapsto {}_H\theta$ are inverse isomorphisms between the lattice of left invariant equivalence relations on G and the lattice of subgroups of G.
(2) If $a \in G$, then the ${}_H\theta$-class of a is aH, which is notation for $\{ah \mid h \in H\}$.
(3) A left invariant equivalence relation on a group is *uniform*: all classes have the same size. This is because the function $\lambda_{ba^{-1}} \colon aH \to bH \colon x \mapsto ba^{-1}x$ is a bijection with inverse $\lambda_{ab^{-1}}$.
(4) If $[G : H]$ represents the number of classes of ${}_H\theta$, then $|G| = [G : H]|H|$. This is called *Lagrange's Theorem*, and it is a consequence of the uniformity of ${}_H\theta$. $[G : H]$ is called the *(left) index of H in G*.

Everything said about left invariant equivalence relations holds for right invariant equivalence relations, but one should replace ${}_H\theta$ with $\theta_H = \{(a, b) \in G^2 \mid ab^{-1} \in H\}$. The fact that $(aH)^{-1} = Ha^{-1}$ can be used to show that the left index of H in G equals the right index of H in G.

Normal subgroups.

We have argued that (i) a kernel of a group homomorphism is an equivalence relation, say θ, that is both left and right invariant, and (ii) left invariant equivalence relations have the form ${}_H\theta$ for $H = 1/\theta$ and right invariant equivalence relations have the form θ_H for $H = 1/\theta$. So a kernel of a group homomorphism must have the form ${}_H\theta = \theta_H$ for some $H \leq G$, and conversely if $H \leq G$ is such that ${}_H\theta = \theta_H$, then θ_H is LI+RI and therefore is the kernel of a group homomorphism.

But ${}_H\theta$ may not equal θ_H for some $H \leq G$. These are equal iff their equivalence classes are equal, and by the second consequence above this happens exactly when $aH = Ha$ for all $a \in G$. We call H a *normal* subgroup of G and write $H \lhd G$ if any of the properties of the next exercise hold for H.

Exercise 5. *The following are equivalent.*

(1) ${}_H\theta = \theta_H$.
(2) $aH = Ha$ *for all* $a \in G$.
(3) $aHa^{-1} = H$ *for all* $a \in G$.
(4) $H = a^{-1}Ha$ *for all* $a \in G$.
(5) $aHa^{-1} \subseteq H$ *for all* $a \in G$.
(6) $H \subseteq a^{-1}Ha$ *for all* $a \in G$.
(7) $(aH)(bH) = abH$ *for all* $a, b \in G$.

If $\theta = \theta_H$ is a kernel of a homomorphism, then the classes of the quotient G/θ (i.e., the classes of θ) are the sets of the form $aH = Ha$. The quotient group operations on $\{aH \mid a \in G\}$ are: $1_{G/\theta} = 1/\theta = H$, $(aH)^{-1} = a^{-1}H$, $(aH)(bH) = abH$.

Exercise 6. *In the sublattice of G, the meet and join of normal subgroups is again normal. Moreover, if at least one of $H, K \leq G$ is normal, then $H \vee K = HK$.*