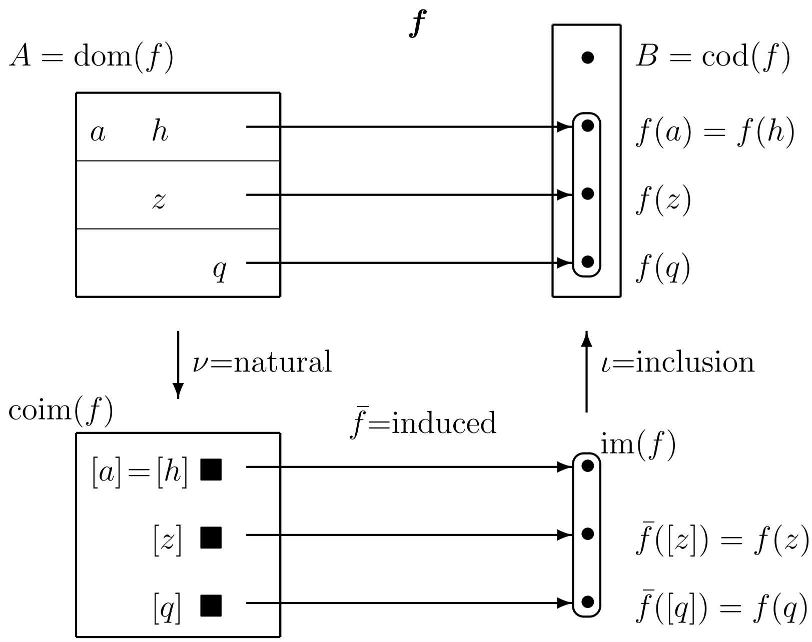
Terminology for functions.

Let A and B be sets and let $f\colon A \to B$ be a function from A to B. There are sets and functions related to A, B and f that have special names.



(1) The *image* of f is $\mathrm{im}(f) = f[A] = \{b \in B \ : \ \exists a \in A(f(a) = b)\}$. The image of a subset $U \subseteq A$ is $f[U] = \{b \in B \ : \ \exists u \in U(f(u) = b)\}$.
(2) The *preimage* or *inverse image* of a subset $V \subseteq B$ is $f^{-1}[V] = \{a \in A \ : \ f(a) \in V\}$.
(3) The preimage of a singleton $\{b\}$ is written $f^{-1}(b)$ and sometimes called the *fiber* of f *over* b. The fiber containing the element a is sometimes written $[a]$.
(4) The *coimage* of f is the set $\mathrm{coim}(f) = \{f^{-1}(b) \ : \ b \in \mathrm{im}(f)\}$ of all nonempty fibers.
(5) The *natural map* is $\nu\colon A \to \mathrm{coim}(f)\colon a \mapsto [a]$.
(6) The *inclusion map* is $\iota\colon\ \mathrm{im}(f) \to B\colon b \mapsto b$.
(7) The *induced map* is $\bar{f}\colon\ \mathrm{coim}(f) \to \mathrm{im}(f)\colon [a] \mapsto f(a)$.

Some facts:

(1) The natural map is *surjective*.
(2) The inclusion map is *injective*.
(3) The induced map is *bijective*.
(4) $f = \iota \circ \bar{f} \circ \nu$. (This is the *canonical factorization* of f.)

Exercises.

(1) Draw a figure like the previous one illustrating $f\colon \mathbb{R} \to \mathbb{R}\colon x \mapsto x^2$. Identify all the "named" sets and functions.
(2) Repeat the previous exercise for the function $g\colon \mathbb{R}^2 \to \mathbb{R}\colon (x, y) \mapsto x + y$.
(3) Repeat for the function $h\colon \{T, F\}^2 \to \{T, F\}\colon (x, y) \mapsto x \vee y$.
(4) Repeat for the *identity function* $id\colon A \to A\colon a \to a$.
(5) Repeat for the *second coordinate projection* $\pi\colon X \times Y \to Y\colon (x, y) \to y$.
(6) Show that
  (a) the composition of two injective functions is injective,
  (b) the composition of two surjective functions is surjective, and
  (c) the composition of two bijective functions is bijective.
(7) Show that injective functions are *left cancellable*: if f is injective, then $f \circ g = f \circ h$ implies $g = h$.
(8) Show that surjective functions are *right cancellable*: if f is surjective, then $g \circ f = h \circ f$ implies $g = h$.
(9) Show that if $f\colon A \to B$ is a function, then $f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A)$ is also a function. Show that f is injective iff f^{-1} is surjective, and f is surjective iff f^{-1} is injective.