ALGEBRA

SUMMARY OF TOPICS FROM 10/19/15-12/11/15

II. Groups. (Continued.)
- (n) Products: projection maps, universal property of products, characterization of products.
- (o) Fundamental Theorem of Finite Abelian Groups, Elementary Divisor form and Invariant Factor form.
- (p) Semidirect products: construction of and characterization theorem.

III. G-sets.
- (a) G-sets, actions of groups, permutation representations of groups: different ways of looking at the same information.
- (b) The left regular G-set is free over a 1-element generating set.
- (c) Orbit-Stabilizer Theorem.
- (d) The size of the conjugacy class of $x \in G$ is $[G\colon C_G(x)]$.
- (e) Not-Burnside's Lemma. (+ applications)
- (f) Cauchy's Theorem.
- (g) $Z(G)$.
- (h) The Class Equation. A nontrivial p-group has a nontrivial center.
- (i) G is abelian iff $G/Z(G)$ is cyclic.
- (j) A group of order p^2 is abelian.
- (k) The Sylow Theorems.
- (l) A description of groups of order pq, $p < q$ prime.
- (m) Simple groups: there is no simple group of order $p^k, k > 1$, pq, p^2q, or pqr.
- (n) Approximate statement of the Classification Theorem for Finite Simple Groups.
- (o) Finite alternating groups. Cauchy number. Even and odd permutations. Simplicity of A_n, $n \geq 5$.
- (p) Projective special linear groups.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.
- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) State the theorem that characterizes products of (two) groups.

(2) How many abelian groups are there of order $360 = 2^3 \cdot 3^2 \cdot 5$ up to isomorphism?

(3) Explain the following fact: the number of vertices, edges or faces of a tetrahedron divides the order of its rotation group. Why is this result false for a pyramid with a square base?

(4) How many ways are there to assign the numbers $1, 2, \ldots, 20$ to the sides of an icosahedron if the sum of numbers on any two opposite faces must be 21?

(5) State the Sylow Theorems.

(6) Show that there is no simple group of order 12.

(7) Show that if P is a Sylow p-subgroup of G, then $n_p = [G\colon N_G(P)]$.

(8) Show that the only group of order $11 \cdot 13 \cdot 17$ is $\mathbb{Z}_{11\cdot 13\cdot 17}$. (Hint: the Sylow theorems are enough to show that $G \cong \mathbb{Z}_{13\cdot 17} \rtimes \mathbb{Z}_{11}$. You need another argument to prove that $\rtimes$ is actually $\times$. This other argument should be based on the fact that $|\mathbb{Z}_{11}| = 11$ and $|\mathrm{Aut}(\mathbb{Z}_{13\cdot 17})| = \phi(13 \cdot 17) = 12 \cdot 16$ are relatively prime.)

(9) What is the sign of the permutation

$$(1)(2\ 3)(4\ 5\ 6)(7\ 8\ 9\ 10)\cdots(106\ \cdots\ 118\ 119\ 120)?$$

(10) It is a fact that the rotation group of the cube is isomorphic to S_4. Explain why all rotations of the cube are even permutations of the vertices. Yet not all elements of S_4 are even permutations. Resolve this apparent contradiction.

(11) Show that A_{n+2} contains a subgroup isomorphic to S_n. Conclude that any finite group G is embeddable in some finite alternating group.