ALGEBRA

SUMMARY OF TOPICS FROM 8/24/15-10/12/15

I. Algebra in general.
   (a) Operations.
   (b) Type of an algebra.
   (c) Homomorphisms and isomorphisms.

II. Groups.
   (a) Definitions of *group* and *abelian group*.
   (b) Properties of group tables. Examples of small groups (up to size 4).
   (c) Cyclic groups.
   (d) Lattice of subgroups.
   (e) Permutations. Cayley Representation Theorem.
   (f) Cycle type of a permutation.
   (g) The groups S_n, C_n, D_n, Q_{4n} (name, notation, elements, order, manner of composition, and presentations).
   (h) Matrix groups $GL_n(\mathbb{R}), SL_n(\mathbb{R}), O(n)$. Every finite group is isomorphic to a group of rotations of finite dimensional Euclidean space.
   (i) Symmetry groups of the Platonic solids. Automorphism groups of algebraic structures.
   (j) Subgroups, normal subgroups, cosets, and quotient groups.
   (k) Lagrange's Theorem, index, and the formula $|G| = [G\colon H]|H|$.
   (l) Conjugacy. Conjugacy classes in S_n are determined by cycle type.
   (m) First, Second and Third Isomorphism Theorems, and the Correspondence Theorem.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Explain why the multiplication table of a group is a Latin square.

(2) Prove or disprove: a group of prime order is cyclic.

(3) Write out the definitions of the following: automorphism, coimage of a function, abelian group, cyclic group, permutation, dihedral group, index of a subgroup.

(4) Describe all homomorphisms $h\colon \mathbb{Z} \to \mathbb{Z}$. What are the kernels and images?

(5) What is the kernel of determinant function $\mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^\times$?

(6) Suppose $\alpha, \beta \in S_n$ and $\beta = (i_1 \ \ldots \ i_k)$. Explain why $\alpha\beta\alpha^{-1} = (\alpha(i_1) \ \ldots \ \alpha(i_k))$.

(7) Suppose H and K are normal subgroups of G. Explain why $H \vee K = HK$.

(8) Suppose G is a finite group, $H \leq G$, and $N \lhd G$. Show that $|HN| = \frac{|H||N|}{|H \cap N|}$.

(9) State the First Isomorphism Theorem.