SET THEORY
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1. Define each object, and give an example.

(a) *Inductive* set.

A set is *inductive* if it contains 0 and is closed under successor. Example: ω.

(b) *Kernel* of a function.

The *kernel* of $f\colon A \to B$ is the binary relation

$$\ker(f) := \{(a,b) \in A \times A \mid f(a) = f(b)\}.$$

Example: the equality relation on A is the kernel of the identity function $\mathrm{id}_A\colon A \to A\colon a \mapsto a$.

(c) *Well-founded* ordered set.

An ordered set $(X,<)$ is *well-founded* if every nonempty subset of X has a minimal element. Example: $(\omega,<)$.

2. Show that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ implies $A \subseteq B$.

Assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$, we derive that $A \in \mathcal{P}(B)$, hence $A \subseteq B$.

3.

(a) What is the recursive definition of exponentiation of natural numbers?

$$\begin{aligned} m^0 &= 1 \\ m^{S(n)} &= m^n \cdot m \end{aligned}$$

(b) Prove that $(m \cdot n)^k = m^k \cdot n^k$ for all $m, n, k \in \mathbb{N}$. (You may use any valid arithmetic results that concern *addition* and *multiplication*, but identify which results you are using.)

Let $P(x)$ be "$\forall m \, \forall n ((m \cdot n)^x = m^x \cdot n^x)$"

$P(0)$:

$$\begin{aligned} (m \cdot n)^0 &= 1 && \text{IC, } \frown \\ &= 1 \cdot 1 && \text{unit law for } \cdot \\ &= m^0 \cdot n^0 && \text{IC, } \frown \end{aligned}$$

$P(k) \to P(S(k))$:

$$\begin{aligned} (m \cdot n)^{S(k)} &= (m \cdot n)^k \cdot (m \cdot n) && \text{RR, } \frown \\ &= (m^k \cdot n^k) \cdot (m \cdot n) && \text{IH} \\ &= (m^k \cdot m) \cdot (n^k \cdot n) && \text{Assoc. and Comm. laws for } \cdot \\ &= m^{S(k)} \cdot n^{S(k)} && \text{RR, } \frown \end{aligned}$$