Restricted quantifiers.

So far we have only discussed two quantifiers, "for all x" and "there exists x". Others are possible, such as "for infinitely many x" or "for all positive x" etc. In this handout I will discuss a broad class of quantifiers that are commonly used in mathematics, called restricted quantifiers. (An important point to keep in mind is that it is never necessary to use them, since anything expressible by restricted quantifiers can also be expressed using the ordinary quantifiers $\forall$ and $\exists$.)

The archetypal restricted quantifier is "for all positive x", which is written $(\forall x > 0)$. This is often used when writing about the real numbers, such as in the sentence

$$(\forall x > 0)(\exists y > 0)(x = y^2),$$

which expresses that every positive number has a positive square root. An expression of the form $(\forall x > 0)Q(x)$ is *by definition* an abbreviation for the expression $\forall x((x > 0) \to Q(x))$, which involves an ordinary quantifier only. Similarly, an expression of the form $(\exists x > 0)Q(x)$ is *by definition* an abbreviation for $\exists x((x > 0) \wedge Q(x))$.

More generally, if $P(x)$ is a predicate, then there are restricted quantifiers $(\forall x P(x))$ and $(\exists x P(x))$ which are used in expressions $(\forall x P(x))Q(x)$ and $(\exists x P(x))Q(x)$ to abbreviate $\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \wedge Q(x))$ respectively. One could read $(\forall x P(x))Q(x)$ as "for all x such that $P(x)$ holds, $Q(x)$ holds."

If quantifiers are used in succession, then the predicates that restrict their range can be more complicated, as in $(\forall x P_1(x))(\exists y \underline{P_2(x, y)})Q(x, y)$, which is an abbreviation for

$$\forall x(P_1(x) \to \exists y(P_2(x, y) \wedge Q(x, y))).$$

A concrete example of a sentence of this type is $(\forall x \neq 0)(\exists y \neq x)(x^2 = y^2)$.

If one replaces $(\forall x P(x))Q(x)$ and $(\exists x P(x))Q(x)$ with their definitions, then one can check that restricted quantifiers behave like ordinary quantifiers:

- $\neg(\exists x P(x))Q(x) \equiv (\forall x P(x))\neg Q(x)$.
- $\neg(\forall x P(x))Q(x) \equiv (\exists x P(x))\neg Q(x)$.
- $((\exists x P(x))Q(x)) \wedge R \equiv (\exists x P(x))(Q(x) \wedge R)$ if x is not free in R.
- ETC.

Also, restricted quantifiers behave in the expected way during quantifier games. If the game is of the form $(\forall x P_1(x))(\exists y P_2(y))Q(x, y)$, then $\forall$ plays first and chooses a value to substitute for x, say $x = a$. If $P_1(a)$ does not hold, then $\forall$ has already lost the game. Otherwise the game continues and $\exists$ chooses (say) $y = b$. If $P_2(b)$ does not hold, then $\exists$ loses immediately. But if the values chosen are "playable", meaning that $P_1(a)$ and $P_2(b)$ hold, then we test $Q(a, b)$ to see if it holds. If it does, then $\exists$ wins, otherwise $\forall$ wins.

One of the most significant appearances of restricted quantifiers in mathematics occurs in the definition of a limit. The expression $\lim_{x \to a} f(x) = L$ is an abbreviation for

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)((0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)).$$

The statement that f is continuous at $x = a$ (i.e., $\lim_{x\to a} f(x) = f(a)$) looks the same with $f(a)$ in place of L:

$$(\forall\epsilon > 0)(\exists\delta > 0)(\forall x)((0 < |x - a| < \delta) \to (|f(x) - f(a)| < \epsilon)).$$

Exercises.

1. Show that any constant function $f : \mathbb{R} \to \mathbb{R}$ is continuous at any point a. (This requires giving a winning strategy for $\exists$.)

2. Show that the signum[1] function is discontinuous at $x = 0$. (Here $\mathrm{sgn}(x) =$ "the sign of x", i.e.,

$$\mathrm{sgn}(x) := \begin{cases} +1 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \text{ and} \\ -1 & \text{if } x < 0. \end{cases}$$

(This requires giving a winning strategy for $\forall$.)

3. Show that $f(x) = x$ is continuous at any a.

4. Modify your answer to Problem 3 so that it works for $f(x) = 2x$ and then modify it again for $f(x) = x/2$.

5. The Axiom of Regularity can be written with restricted quantifiers as:

$$(\forall x \neq \emptyset)(\exists y \in x)(x \cap y = \emptyset).$$

Rewrite it so that it does not use restricted quantifiers.

[1] "Signum" is Latin for "sign". Sometimes the signum function is called the sign function, but do not confuse it with the sine function.