DISCRETE MATH
MIDTERM

Name:______________________________

You have 50 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

1.

(a) Rewrite the following sentence as a formal proposition.

If I eat grapes, then I will not eat bananas, and if I eat bananas, then I will not eat pomegranates, and if I eat pomegranates, then I will not eat grapes, but I will surely eat either grapes, bananas or pomegranates.

Let G="I eat grapes", B="I eat bananas", P="I eat pomegranates".

Answer = $(G \to (\neg B)) \wedge (B \to (\neg P)) \wedge (P \to (\neg G)) \wedge (G \vee B \vee P)$

(b) Write a truth table for the proposition. Is it a tautology?

G	B	P	$W = G \to (\neg B)$	$X = B \to (\neg P)$	$Y = P \to (\neg G)$	$Z = G \vee B \vee P$	$W \wedge X \wedge Y \wedge Z$
T	T	T	F	F	F	T	F
T	T	F	F	T	T	T	F
T	F	T	T	T	F	T	F
T	F	F	T	T	T	T	T
F	T	T	T	F	T	T	F
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	F	F

It is not a tautology.

2. Show that $A \subseteq B$ implies $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Assume that $A \subseteq B$. We must show that any $X \in \mathcal{P}(A)$ satisfies $X \in \mathcal{P}(B)$. Equivalently, we must show that $X \subseteq A$ and $A \subseteq B$ together imply $X \subseteq B$. (I.e. $X \subseteq A \subseteq B$ implies $X \subseteq B$.)

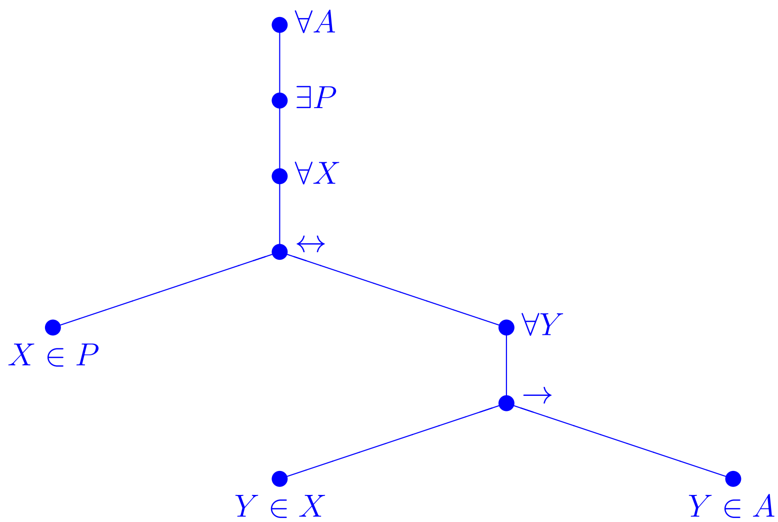
For any $Y \in X$ we have $Y \in A$ (since $X \subseteq A$) and therefore $Y \in B$ (since $A \subseteq B$). Hence $Y \in X$ does imply $Y \in B$, yielding $X \subseteq B$.

3.

(a) Write the axiom of power set as a formal sentence.

$$\forall A\, \exists P\, \forall X((X \in P) \leftrightarrow \forall Y((Y \in X) \to (Y \in A)))$$

(b) Draw a formula tree for the sentence from (a).



4. Consider the sentence $\forall x((0 < x) \to \exists y(y^2 = x))$.

   (a) Describe a winning strategy for some quantifier that decides the truth of this sentence in $(\mathbb{R}, \{+, -, 0, \cdot, 1, <\})$.

   A prenex form for this sentence is $\forall x\ \exists y((0 < x) \to (y^2 = x))$.

   The sentence is true in $(\mathbb{R}, \{+, -, 0, \cdot, 1, <\})$. A winning strategy for $\exists$ is: let $\forall$ choose a value for x. If the value is not positive, then $\forall$ has already lost, so $\exists$ may choose any value for y. (For definiteness, let $\exists$ choose $y = 0$ in this case.) If $\forall$ chooses a positive value for x, then $\exists$ should choose y so that $y^2 = x$ is true. This is possible since every positive number in $\mathbb{R}$ is a square in $\mathbb{R}$. With these choices $0 < x$ is true and $y^2 = x$, so $((0 < x) \to (y^2 = x))$ is true.

   (b) Describe a winning strategy for some quantifier that decides the truth of this sentence in $(\mathbb{Z}, \{+, -, 0, \cdot, 1, <\})$.

   The sentence $\forall x\ \exists y((0 < x) \to (y^2 = x))$ is false in $(\mathbb{Z}, \{+, -, 0, \cdot, 1, <\})$. A winning strategy for $\forall$ is: Choose $x = 2$. Then $0 < x$ is true, and there is no value for y in $\mathbb{Z}$ that makes $y^2 = x$ true. Hence $((0 < x) \to (y^2 = x))$ is false.